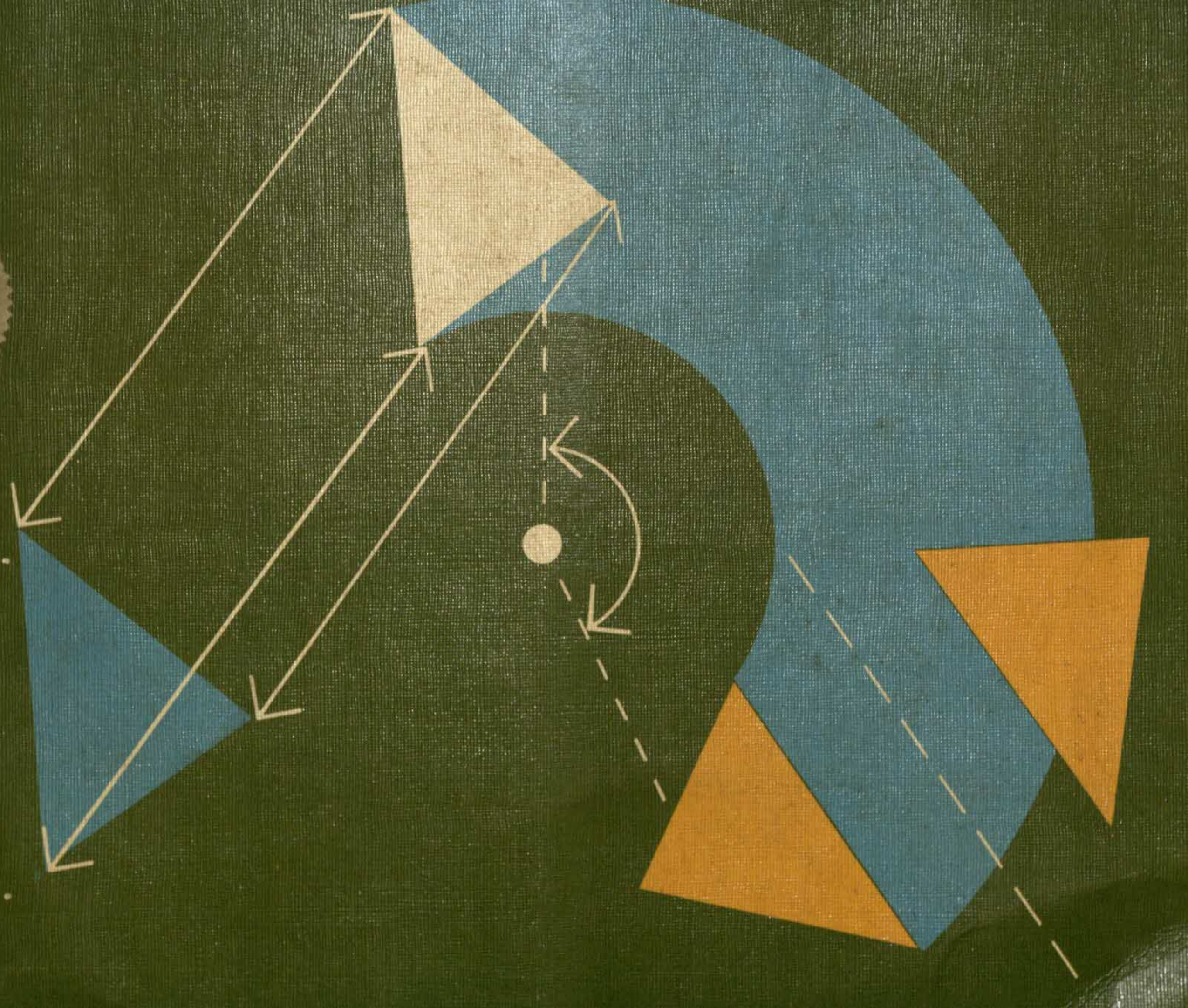
PENGUIN EDUCATION

Learning Mathematics
Book Two

Raymond S. Heritage

The Shropshire Mathematics Experiment



The Shropshire Mathematics Experiment   Penguin Books

Learning Mathematics
Book Two

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Section 1 **Working with decimal fractions**

Exercise 1 **a**

i What is $2\frac{1}{2}$ in decimals?
ii What is the sum of $2\frac{1}{2}$ and $2\frac{1}{2}$?
iii What is $3\frac{1}{4}$ as a decimal?
iv $3\frac{1}{4}+3\frac{1}{4} =$
v What is $5\frac{3}{4}$ as a decimal?
vi Find the sum of $5\frac{3}{4}$ and $8\frac{1}{4}$.
vii What is $8\frac{1}{4}$ as a decimal?
viii What is $7\frac{3}{8}$ as a decimal?
ix What is $9\frac{1}{8}$ as a decimal?
x Add $7\frac{3}{8}$ to $9\frac{1}{8}$
xi Convert $12\frac{5}{8}$ to a decimal.

b

i Add 2·5 to 2·5
ii $3{\cdot}25+3{\cdot}25 =$
iii $5{\cdot}75+8{\cdot}25 =$
iv $7{\cdot}375+9{\cdot}125 =$
v $9{\cdot}125+12{\cdot}625 =$

c

i Look at $25+25 = 50$ and $2{\cdot}5+2{\cdot}5 = 5{\cdot}0$
Can you now add 3·5 to 3·5?

ii Look at $325+325 = 650$ and $3{\cdot}25+3{\cdot}25 = 6{\cdot}50$
Can you now add 7·25 to 7·25?

iii Compare $575+825 = 1400$ with the result of $5{\cdot}75+8{\cdot}25$
Can you now add 6·75 and 12·25?

iv Compare $7375+9125$ with $7{\cdot}375+9{\cdot}125$
Now add 5·275 and 8·675

v Add 9675 and 2875
Now add 9·675 to 2·875

By comparing the addition of decimals with that of ordinary numbers you have discovered the main principle of working with decimals.

We will now lay the additions out the other way. Compare the operations on natural numbers and decimal fractions.

$$\begin{array}{r} 6834 \\ +\ 2925 \\ \hline 9759 \end{array} \qquad \begin{array}{r} 6{\cdot}834 \\ +\ 2{\cdot}925 \\ \hline 9{\cdot}759 \end{array} \qquad \begin{array}{r} 7864 \\ +\ 8430 \\ \hline 16294 \end{array} \qquad \begin{array}{r} 7{\cdot}864 \\ +\ 8{\cdot}43 \\ \hline 16{\cdot}294 \end{array}$$

Here are some subtraction calculations. Compare again.

$$\begin{array}{r} 6839 \\ -\ 3427 \\ \hline 3412 \end{array} \qquad \begin{array}{r} 6{\cdot}839 \\ -\ 3{\cdot}427 \\ \hline 3{\cdot}412 \end{array} \qquad \begin{array}{r} 12458 \\ -\ 9339 \\ \hline 3119 \end{array} \qquad \begin{array}{r} 12{\cdot}458 \\ -\ 9{\cdot}339 \\ \hline 3{\cdot}119 \end{array}$$

$$\begin{array}{rrrrrrrr} & 17084 & & 17{\cdot}084 & & 10004 & & 10{\cdot}004 \\ - & 6493 & - & 6{\cdot}493 & - & 2168 & - & 2{\cdot}168 \\ \hline & 10591 & & 10{\cdot}591 & & 7836 & & 7{\cdot}836 \end{array}$$

Compare these calculations too.

$$\begin{array}{rrrrrrrr} & 15430 & & 15{\cdot}430 & & 154{\cdot}30 & & 1{\cdot}5430 \\ + & 1700 & + & 1{\cdot}700 & + & 17{\cdot}00 & + & 0{\cdot}1700 \\ \hline & 17130 & & 17{\cdot}130 & & 171{\cdot}30 & & 1{\cdot}7130 \end{array}$$

Now compare this set of calculations.

$$\begin{array}{rrrrrrrr} & 12410 & & 124{\cdot}10 & & 1{\cdot}2410 & & 12{\cdot}41 \\ - & 3271 & - & 32{\cdot}71 & - & 0{\cdot}3271 & - & 3{\cdot}271 \\ \hline & 9139 & & 91{\cdot}39 & & 0{\cdot}9139 & & 9{\cdot}139 \end{array}$$

Exercise 2 Work out the following calculations in decimals.

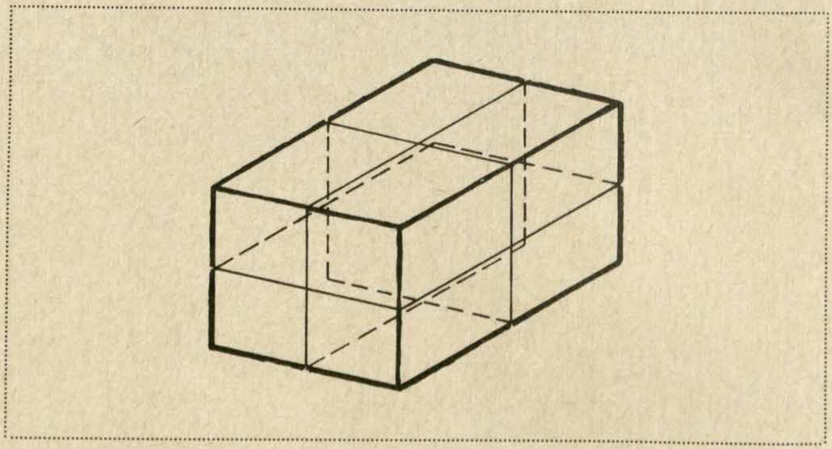
a First add the numbers in each part; then subtract the second number from the first.

i 45·62, 38·59
ii 45·62, 3·859
iii 10·07, 1·64
iv 10·07, 0·164
v 41·59, 8·702
vi 8·702, 4·159
vii 2·003, 1·954
viii 80·69, 35·06

b Simplify the following:

i $27{\cdot}63+4{\cdot}85-9{\cdot}763$
ii $178{\cdot}39-64{\cdot}06+19{\cdot}73$
iii $178{\cdot}39-(64{\cdot}06+19{\cdot}73)$
iv $347-8{\cdot}9-12{\cdot}3$
v $26{\cdot}78-31{\cdot}67+25{\cdot}54$

Exercise 3 **a**



A packing case is 36·5 in. long, 24·25 in. wide and 12·75 in. high. It is bound as in the diagram with wires, using three separate pieces. The ends are fixed together by seals. When the seals are cut, how long are the three pieces of wire which come off, provided no wire is taken with the seals?

b A machine is cutting lengths of metal strip automatically. It cuts accurately to within five hundredths of an inch. This means that if it is set to cut lengths of 15·5 in. it may cut five hundredths of an inch bigger or smaller than 15·5 in. What are the longest and shortest lengths it will cut when set like this?

c Jim's uncle wants to send him some books for his birthday. He has bought books in a series which are all the same length and width but different thicknesses. He has also a box big enough to take the length and width but not deep enough to hold all the books. The box is 10 cm. deep. The books he has are (in centimetres) 2·75, 1·5, 0·5, 1·35, 1·75, 2·85, 2·05, 1·05, 2·65 thick. What is the best combination of books for Jim's uncle to send if he wants to send the greatest number of different books?

Exercise 4 Work out these calculations and identify the answers: they are given, but in the wrong order.

a
i $1{\cdot}932+4{\cdot}960+3{\cdot}873$
ii $73{\cdot}4+92{\cdot}53+3{\cdot}91$
iii $6{\cdot}103+8{\cdot}735+2{\cdot}343$
iv $3{\cdot}725+1{\cdot}038-1{\cdot}437$
v $92{\cdot}53-78{\cdot}3+16{\cdot}94$
vi $527{\cdot}3-(41{\cdot}2+239)$

Answers: 169·84, 10·765, 247·1, 3·326, 17·181, 31·17.

b
i $6{\cdot}59-1{\cdot}282$
ii $22{\cdot}34-0{\cdot}48$
iii $5{\cdot}196-0{\cdot}4034$
iv $9{\cdot}216-1{\cdot}12$
v $4{\cdot}069-1{\cdot}58$
vi $40{\cdot}43-22{\cdot}821$

Answers: 2·489, 17·609, 5·308, 8·096, 4·7926, 21·86

Section 2 **Sets again**

Look at the following sets and the statements underneath them. Study them carefully and decide what the new symbol means.

$$A = \{12, 17, 24, 97, 103\}$$
$$B = \{13, 18, 25, 98, 104\}$$
$$C = \{14, 19, 26, 99, 105\}$$

$12 \in A$; $24 \in A$; $25 \in B$; $19 \in C$; $104 \in B$

Exercise 1 **a** Complete the following:

$17 \in \quad ; 18 \in \quad ; 105 \in \quad ; 97 \in$

b What is another way of saying 'a member of a set'? What do you think the symbol '$\in$' stands for?

c What is $n(A)$ equal to?

d How can you show without counting that $n(A) = n(B)$?

e Is $n(B) = n(C)$?

f What do you call the process you are using?

g If M, N, and P are three sets and $n(M) = n(N)$ and $n(N) = n(P)$, what can you say about $n(M)$ and $n(P)$? Would it also be true to say that $n(M) + n(P) = 2 \, . \, n(N)$?

h $X = \{x \mid 20 < x \leqslant 30;\ x \text{ is a multiple of } 2\}$
$Y = \{y \mid 50 \leqslant y < 60;\ y \text{ is even}\}$

What can you say about $n(X)$ and $n(Y)$?

Four boys, John Smith, Owen Jones, Billy Brown, John Bull, are the winners in a race. John Smith wins first prize, John Bull second, Owen Jones third, and Billy Brown fourth. If we arrange the boys as a set and the prizes as another set, we can set up a one-to-one correspondence or mapping between them.

{John Smith, John Bull, Owen Jones, Billy Brown}
↕ ↕ ↕ ↕
{1st prize, 2nd prize, 3rd prize, 4th prize}

Now, John Smith's father owns a car, Owen Jones' father has two cars, Billy Brown's father runs a taxi service, and John Bull's father has no car. Can you set up a one-to-one correspondence between the fathers and their cars? Of course not; the relation is many-to-one.

Exercise 2 Which of the following are one-to-one relations and which many-to-one?

a boy–mother

b vertices of a square and the points of the compass

c brother–sister

d husband–wife

e you and your shoes

f you and your form-room

g months and Wednesdays

h years and April Fool Days

i $\{1, 3, 5, 7\}$ and $\{0, 2, 4, 6\}$

j set A and set B where $n(A) = n(B)$

Instead of saying the long phrase 'setting up a one-to-one mapping between' we often simply say 'mapping onto'. Earlier we mapped the boys onto the prizes or, alternatively, the prizes onto the boys.

Exercise 3

$$\begin{aligned} N_1 &= \{0, 1, 2, 3, 4, 5, \ldots\} \\ N_2 &= \{0, 2, 4, 6, 8, 10, \ldots\} \\ N_3 &= \{0, 4, 8, 12, 16, 20, \ldots\} \end{aligned}$$

a Is N_2 a subset of N_1? Is $N_3 \subset N_2$?

b Can you always find an element of N_1 to match each element of N_2 and vice versa?

c Can you map N_1 onto N_2?

d Can you map N_2 onto N_3?

e What can you say about $n(N_1)$, $n(N_2)$, and $n(N_3)$?

f How many elements are there in set N_1?

If two sets can be mapped onto one another and their elements can be paired off with none left over in either set, the sets are then equivalent sets. We can use the symbol ↔ for 'is equivalent to'. This kind of mapping is also known as a *bijection*.

Exercise 4 Write down the equivalences in the following. (The first is done for you.)

a $A = \{a, e, i, o, u\}$ $B = \{1, 2, 3, 4\}$ $C = \{♭, ♯, ♮, ×, 𝄫\}$
$A \leftrightarrow C$

b $D = \{0, 2, 4, 6, 8\}$ $E = \{1, 3, 5, 7, 9\}$
$F = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

c $G = \{x \mid 25 < x < 35; x \text{ is even}\}$ $H = \{y \mid 106 < y \leqslant 115; y \text{ is odd}\}$
$J = \{z \mid 89 \leqslant z < 99; z \text{ is odd}\}$

d $K = \{0, 1, 2, 3, \ldots\}$ $L = \{0, 2, 4, 6, 8 \ldots\}$
$M = \{1, 3, 5, 7, 9 \ldots\}$

e $N = \{\text{points in a straight segment AB, 3 in. long}\}$
$M = \{\text{points in a straight segment CD, 7 in. long}\}$

A symbol which you have met before is: $\Leftrightarrow$.
You wrote this sign when you were solving equations.
We are now going to think about it again.
Think about the two statements:

A: It is daylight. B: The sun has risen.

Now it is obvious that

C: If it is daylight, the sun has risen.

Is D also true?

D: If the sun has risen, it is daylight.

No, surely not; there may be a lunar eclipse of the sun, when it is not daylight, but the sun has risen. On thousands of days when the sun has risen, it is daylight. If there is one example of a day when this is not so, it is enough to deny the statement D. However, statement C is always true. We can therefore say:

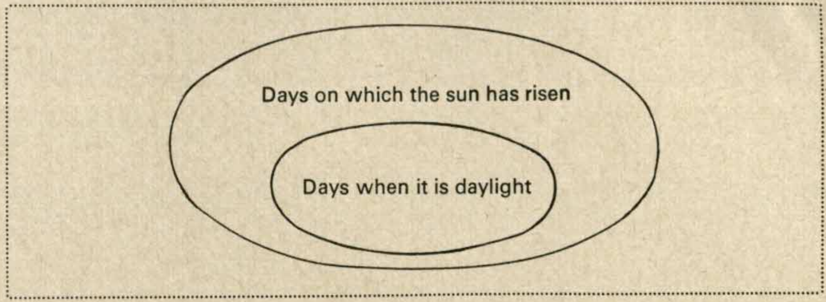
'It is daylight' implies that 'the sun has risen' (statement C)
or $A \Rightarrow B$

On the other hand

$B \Rightarrow A$ (statement D)
or $A \Leftarrow B$ (statement D)

is not true.

We can even draw a diagram to explain this example. This diagram is a Venn diagram, about which we shall learn a good deal more later on.



The size and shape of the curves does not matter, but the correct one must lie wholly inside the other.
It follows that:

Days when it is daylight $\subset$ Days when the sun has risen.

Exercise 5 In each of the following say which of these is true: $A \Rightarrow B$, $B \Rightarrow A$, $A \Leftarrow B$.

a Statement A: I travel from Wellington to Shrewsbury by car.
Statement B: I motor 12 miles.

b A: The Wrekin is a 1334 ft. high hill in Shropshire.
B: There is at least one hill over 1300 ft. high in Shropshire.

c A: All the hotels in Calle Hortaleza, Madrid are called 'Ducal'.
B: Hotel Ducal is in Calle Hortaleza, Madrid.
(Do you know if either of the statements themselves is true? Does it in fact matter for the question whether or not they are true?)

d A: This rath is a mome.
B: All raths are momes.
(Does it matter for the question that you have no idea what the statements mean?)

e A: Carbon dioxide turns lime-water milky.
B: This gas, which has turned lime-water milky, must be carbon dioxide.

You could write down statements of a slightly different kind, for instance:

A: John Smith is Mary Smith's father.

B: Mary Smith is John Smith's daughter.

Then $A \Rightarrow B$ and also $A \Leftarrow B$. We can write this down in a single statement as $A \Leftrightarrow B$.

Exercise 6 Check the following:

a A: Joan is Mary's cousin. B: Mary is Joan's cousin.
$A \Leftrightarrow B$

b A: 'Chalky' is Willy Smith's teacher.
B: Willy Smith is 'Chalky's' pupil.
$A \Leftrightarrow B$

c A: $x-4 = 12$ B: $x = 16$
$A \Leftrightarrow B$

d A: $2x-5 = 15$ B: $x = 10$
$A \Leftrightarrow B$

e A: $3x = 5$ B: $x = 15$
$A \Leftrightarrow B$

f A: $15x-4x+3 = 14$ B: $x = \frac{17}{11}$
$A \Leftrightarrow B$

g $\dfrac{5x-7}{8} = 6 \Leftrightarrow x = 11$

h $\dfrac{7x-5}{9} = 1 \Leftrightarrow x = 2$

i $\dfrac{7(x-5)}{3} = 4 \Leftrightarrow x = 14\frac{3}{7}$

j $5(x-6)+2(x+8) = 23 \Leftrightarrow x = 3$

k $5(x+6)+2(x-8) = 19 \Leftrightarrow x = 2$

Exercise 7 Complete the following implications:

a $\dfrac{x-4}{2} = 7 \Leftrightarrow x =$

b $\dfrac{x-6}{5} = 5\frac{1}{5} \Leftrightarrow x =$

c $2x+7 = 13 \Leftrightarrow x =$

d $2(x+7) = 14 \Leftrightarrow x =$

e $2-x = 1 \Leftrightarrow x =$

f $12-x = 5 \Leftrightarrow x =$

g $\dfrac{14-x}{3} = 3 \Leftrightarrow x =$

h $\frac{1}{3}x+5 = 12 \Leftrightarrow x =$

i $27-\dfrac{2x}{3} = 7 \Leftrightarrow x =$

Exercise 8 Say whether the following are true or false, where A and B are sets:

a $A \leftrightarrow B \Leftrightarrow n(A) = n(B)$

b $A \leftrightarrow B$ and $A \cap B = \phi \Leftrightarrow 2.n(A) = n(A \cup B)$

Exercise 9 **a** An even number is defined as the product of any natural number and 2. From this definition, would you say that zero is an even number?

b i How could you define a multiple of 3 so that zero is a multiple of 3?
ii Is zero divisible by 3?
iii Are the non-zero multiples of 3 divisible by 3?

c Are sets of multiples finite or infinite?

d i Write down the set A of non-zero multiples of 2 less than 22.
ii Write down the set B of non-zero multiples of 3 less than 21.
iii Write down the set of common (non-zero) multiples of 2 and 3.
iv How would you describe this set in terms of A and B? Why?
v Is this set an ordered set?
vi What is its least member and how is this number related to 2 and 3?

e Repeat **d** with 4 and 6 instead of 2 and 3. Answer (vi) very carefully.

Answers
Exercise 7 **a** 18 **c** 3 **e** 1 **g** 5

Section 3 **Using decimals**

Exercise 1 **a** What kind of units are usually used for the measuring of area?

b Give suitable units for measuring (i) the area of a classroom wall, (ii) the area of a house window, (iii) the area of the surface of an electric radiator, (iv) the area of plastic used for backing a book, (v) the area of grass on a cricket field, and (vi) the area of curtains in a school hall.

c Jane has taken up photography as a hobby, and she is selling prints at a rate of 0·15 pence per square in. Find what she charges for

i postcard size prints ($3\frac{1}{2}$ in. by $5\frac{1}{2}$ in.)
ii quarter plate prints ($3\frac{1}{2}$ in. by $4\frac{1}{2}$ in.)
iii whole plate prints.

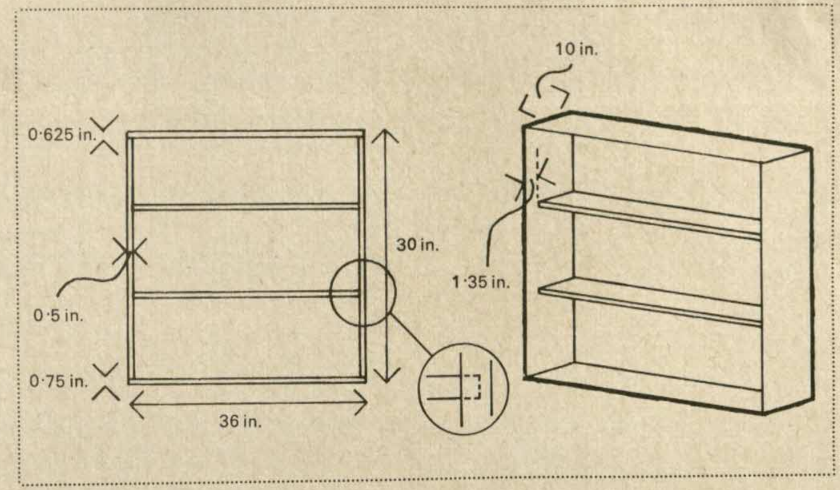
What size will whole plate prints be, assuming they are a proper multiple of quarter-plate?

d Jane went on holiday to stay with her French pen friend in Niort. When she got there she managed to borrow some photographic equipment and wanted to make some postcard prints of her friend's family. She could not remember the sizes of paper she needed to buy. She knew that it was 9 cm. wide and she could remember that the area was 135 sq. cm. This was enough for her. She went to the shop and bought the paper she needed. What size was it?

e There is another size of paper 9 cm. wide and of area 103·5 sq. cm. How long is it?

f I have just bought a paperbacked book which I want to use a good deal. I intended to cover it with adhesive plastic sheet but when I got my box of sheets out I found that I had left only two pieces, one 16 cm. by 19 cm. and another 9 cm. by 20 cm. The front cover of book measures 11·2 cm. by 18·2 cm. while the book is 1·3 cm. thick. By checking areas I have discovered whether I have enough plastic. Have I? If you think I have, then work out how I can cover the book with it, and how big the 'turn over' at the edges will be. If I did not have any turn over at the edges, how big would the smallest single sheet of plastic have to be to cover the book?

g Jim is going to make a bookcase in his spare time. Two views of it are shown in his two sketches. You will notice that he has given the sizes in decimals of an inch.



He has to make a list of the wood he requires in which the sizes are to be given as common fractions. Make such a list for him not including the shelves or the back. For the shelves he has already a board 9 in. wide and $72\frac{1}{2}$ in. long. Is it big enough to make the shelves? (Remember that a saw cut uses about an eighth of an inch of wood.) He will use hardboard for the back and it is to be rebated $\frac{1}{4}$ in. all round. How big will the board have to be?

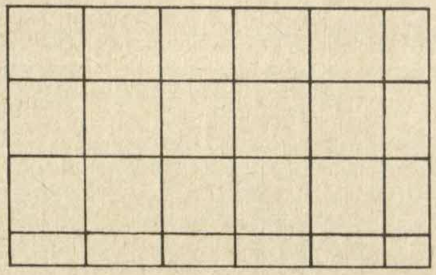
h Think carefully about the following statements which are correctly given:

$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$; $0{\cdot}5\times0{\cdot}5=0{\cdot}25$; $\frac{1}{2}\times5=2\frac{1}{2}$; $0{\cdot}5\times5=2{\cdot}5$; $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$; $\frac{1}{8}=0{\cdot}125$; $0{\cdot}5\times0{\cdot}25=0{\cdot}125$

Now answer the following questions:

- i What is the connexion between the numbers 5, 5 and 25?
- ii What is the connexion between the numbers 5, 25, and 125?
- iii In the calculation of $0{\cdot}5\times0{\cdot}5$, how many digits follow decimal points on the *left* of the equals sign?
- iv In the same calculation, how many digits follow the decimal point on the *right* of the equals sign?
- v Answer iii and iv again for the statement beginning $0{\cdot}5\times5$.
- vi Convert $\frac{1}{4}$ to decimals.
- vii Work out $\frac{1}{4}\times\frac{1}{4}$ in common fractions and then in decimals.
- viii What is $\frac{1}{16}$ as a decimal?
- ix Work out $\frac{1}{2}\times\frac{1}{8}$ in both common and decimal fractions.
- x Work out $0{\cdot}375\times0{\cdot}5$. What is the corresponding calculation in common fractions?

Exercise 2



The sketch shows a rectangle 5·6 in. long and 3·4 in. wide.

a If the rectangle is tessellated in squares one tenth of an inch long, how many such squares are there? Explain how you obtained the answer.

b How many of the small squares are there in one square inch?

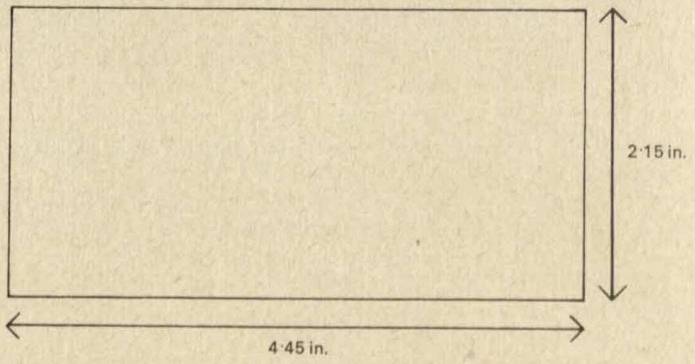
c What is the area of the rectangle in square inches?

d What is the connexion between the calculation you have just done and the following:

$$\frac{56}{10} \times \frac{34}{10} = \frac{1904}{100} \qquad ?$$

What do you think is the significance of the 100?

Exercise 3 **a**



How many squares one hundredth of an inch long are there if you tessellate this rectangle with them?

b How many such small squares are there in one square inch?

c What is the area of the rectangle in square inches?

d Compare your calculation with:

$$\frac{445}{100}\times\frac{215}{100}=\frac{\qquad}{10{,}000}.$$

Complete this calculation and explain the meaning of the denominators of 100 and 10,000.

e Can you see a method of multiplying 4·45 and 2·15 without converting them to common fractions? How do you know where to put the decimal point in the answer?

Exercise 4 A space traveller landed on a planet Kopnkia in another solar system. Near where he landed, there was a deserted building which he entered. It appeared to be a sort of school-room and a blackboard was covered with strange symbols. There were only ten different signs apart from some dots, some × signs and some = signs. The traveller guessed that what he was looking at was arithmetic worked to the base ten. He concluded that the Kopnkians had discovered decimals. There were also exercise-books containing calculations. He looked at one and came to the conclusion that some of the calculations were wrong. Here they are:

i ▽·□ × ⊛ = ▽̊·▽
ii ▽·□ × ⊛·△ = ▽̊·△̊□
iii ⊛·□△ × ▽·⊛ = ▽̊⊛̊·□̊▽̊△
iv ▽·△̊🜁 × ▽·△ = 🜁🜁·△̊🜁
v □·▽̊△ × ○·⊛ = △·□△⊛

If he were right in thinking these were decimal multiplications, which ones were wrong?

After a lot of guessing, the traveller decided he could translate the symbols thus:

0	1	2	4	5	6	7	9
○	△	▽	□	🜁	△̊	▽̊	□̊

See if you can discover which symbols stand for 3 and 8.

Exercise 5 Look carefully at the following calculations and then do the decimal multiplications which follow.

$$\begin{array}{r} 3{\cdot}6 \\ \times\quad 7{\cdot}0 \\ \hline 25{\cdot}2 \end{array}\qquad\qquad \begin{array}{r} 7{\cdot}6 \\ \times\quad 2{\cdot}1 \\ \hline 15{\cdot}2 \\ 0{\cdot}76 \\ \hline 15{\cdot}96 \end{array}\qquad\qquad \begin{array}{r} 6{\cdot}21 \\ \times\quad 2{\cdot}34 \\ \hline 12{\cdot}42 \\ 1{\cdot}863 \\ 0{\cdot}2484 \\ \hline 14{\cdot}5314 \end{array}$$

a 7·2 × 9·0

b 9·2 × 7·3

c 12·4 × 0·56

d 12·6 × 2·84

e 127·2 × 22·5

f 129·4 × 37·2

g 6·54 × 7·34

h 7·82 × 6·23

i 15·234 × 17·456

Exercise 6 **a** A flat electric radiator is 47·5 cm. high and 165 cm. long. What is the surface area of the front? What is the area to the nearest 1000 sq. cm.? If the power of the heater is 1600 watts. and the heater element is distributed evenly all over the heater, about how many sq. cm. are there per watt? What would be the approximate wattage of a heater of similar design but 91·5 cm. high and 150 cm. long?

b Sally knocked her tennis ball through a window at home the other day. The window was in a steel frame and measured 48·5 cm. by 109·75 cm. A new sheet of glass will cost 2 shillings per 1000 sq. cm. and Sally has agreed to pay for the glass at the rate of 1s. 6d. each week. How much will the glass cost to the nearest sixpence and for how long will Sally be contributing her pocket money to the cost?

c The best petrol in Austria costs 3·7 schilling per litre. There are 4½ litres to an imperial gallon. How many schillings does one gallon cost? 1 schilling is about 3¼d. What is the approximate cost of a gallon of Austrian petrol in English currency?

d In Greek currency 1 lepta is 0·01 of 1 drachma.

i On a Greek hotel wine list, the best wine is listed at 40 drachmas. The purchaser has to pay 15% service charge and 5% local tax, both percentages being calculated on the list price. How much will the wine cost him? About how much is this in sterling if there are approximately 83 drachmas to £1?

ii At another hotel the charges are as follows (all prices being in drachmas): breakfast 15, lunch and dinner 50 each, room 120. To this there has to be added a 15% service charge (added to the total bill.) If the charge is rounded up to the nearest drachma, find the total cost of one day's stay. By how many leptas is the traveller being overcharged by the rounding up? Would you say that this overcharge matters very much?

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Section 4 **Reflection in geometry**

For this section, you will need a quantity of paper (any sort will do provided it is not too flimsy), a pair of scissors, a pin and a mirror.

Exercise 1 **a** Take a piece of paper of any shape and fold it, making the crease sharp with your finger-nail. Open it out. What sort of line is on the paper? Is it possible to get a curved line this way?

b Refold the paper and push the pin anywhere through the two thicknesses. Open out. What do you notice about the two holes you have made?

c Place the mirror along the crease and look into it. What do you see?

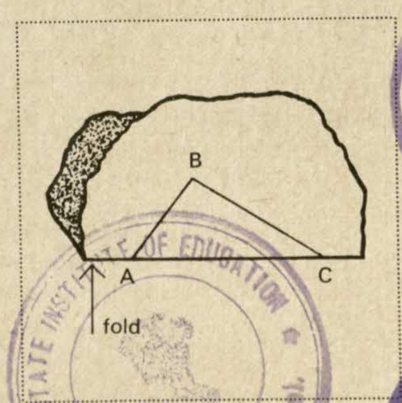
d Fold the paper so that the new crease passes through the two pin-holes. Open out the paper. What can you say about the two creases?

e Place the mirror along one of the creases and look into it. What do you notice about the other crease and its reflection?

f Refold the paper along both creases. What is the angle formed? What do you notice about the two halves of the crease which is folded back on to itself?

Exercise 2 **a** Fold another sheet of paper and cut along two lines AB and BC. Make sure they are different lengths in the first instance. By using different positions for C, you can obtain altogether four different shapes. Cut these shapes and open them out.

b What do you notice about the two triangles divided by the crease? Are they equal in area? Are they congruent? Are they reflections of one another?



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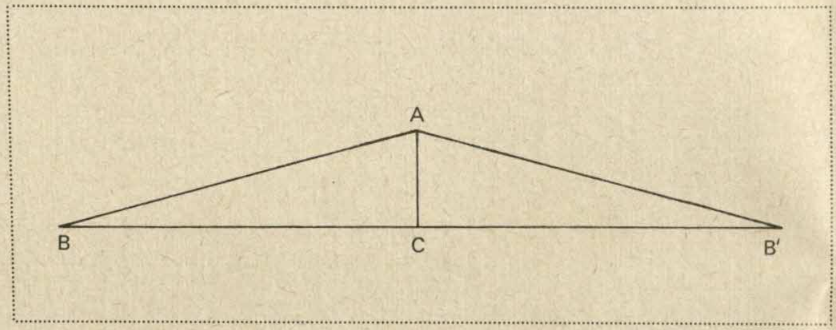
One of your four shapes is a triangle. The others have four sides and are called QUADRILATERALS. You should have a kite-shape, a diamond and a chevron. We are going to examine these shapes in turn, starting with the triangle.

Exercise 3



What can you say about the following?

a the lengths AB and AB′

b angles ABC and AB′C

c triangles ABC and AB′C

d the line AC and the line BB′

e BC and CB′

f the line AC and the angle BAB′

These are the properties of a special sort of triangle. It is called an ISOSCELES TRIANGLE. It is only necessary for us to say that a triangle has two equal sides for us to know that it is isosceles. The word 'isosceles' means 'equal legs': think of a stick-man.

Exercise 4 **a** Examine the room you are in, your house, your school and so on and find examples of isosceles triangles. (A start can be: scissors, pair of compasses, set square...)

b Could you tessellate an infinite plane with congruent, isosceles triangles? Are they all the same way up? Illustrate how you have done it and explain why the arrangement of triangles gives parallel sided strips.

Exercise 5 Examine your kite-shape. In the diagram the dotted line is the crease.

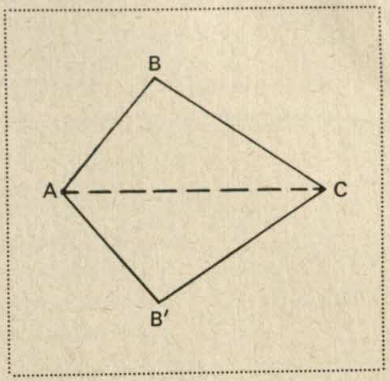
a Why is B′ so called?

b If you place a mirror along AC and reflect triangle ABC in it what do you see? Is this confirmed by folding the paper along the crease?

c What do you conclude about triangle ABC and triangle AB′C?

d Write down sides or angles equal to the following:

i AB
iii B′C
ii angle BCA
iv angle B′AC.



e Fold your kite along the line BB′. As you do it, what happens to the two parts of the crease AC?

f Open the kite out again. What can you say about the angles between the creases?

g Check this statement to see if you agree with it or not. (The lines joining opposite vertices or corners are called DIAGONALS.) 'A kite is a quadrilateral with two pairs of equal adjacent sides; the diagonals meet at right angles; one diagonal divides the kite into two congruent triangles and bisects the angles between the equal pairs of sides.'

Exercise 6 Your diamond shape is already folded along one diagonal. Fold it now along the other diagonal and open it out again. Say which, if any, of the following statements are true.

a The diagonals bisect each other.

b The diagonals are not at right angles.

c The diagonals divide the quadrilateral into four congruent triangles.

d Any pair of adjacent congruent triangles is a reflection (in the diagonal) of the other two.

e The four sides are not equal.

f The four angles are equal.

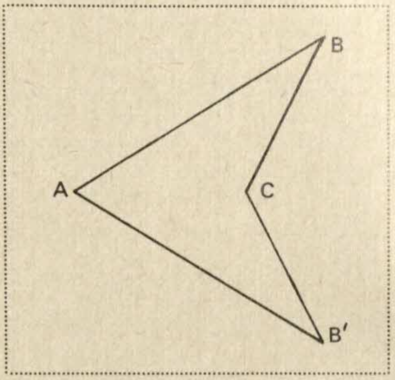
g Opposite angles are equal.

h The four triangles are isosceles.

i There are four isosceles triangles in the figure.

This diamond shape has the special name RHOMBUS. We shall meet it again shortly.

Exercise 7 Look at the chevron or arrow-head.



a What do you notice about the position of one diagonal?

b Place a ruler edge in the position of the diagonal BB′. What do you notice about the angles between the diagonals?

c Make a list of all the properties you can find in this figure.

d Compare the angles of the kite and rhombus and chevron. Observe which are acute and which not. What is special about the chevron?

Exercise 8 Say whether or not you can tessellate an area with the shapes listed. (You may cut out extra shapes to try it if you wish.) In each case try to say why you give these answers.

a rhombus

b kite

c chevron

d any triangle

Exercise 9 Imagine you are given a piece of paper with a single straight fold across it. On the paper there is also a point marked. Show how you can fold the paper to give a second line passing through the point and at right angles to the first fold. You have two cases to consider. In one the point is on the first fold, and in the other it is not. Are these two cases really different? Two lines such as you have just 'drawn' at right angles to each other, are said to be *perpendicular to each other* or *mutually perpendicular*. Some people think that 'perpendicular' is the same as 'vertical'. Do you?

Section 5 **More about decimals**

Exercise 1 **a** A rectangle is 2·75 in. long and 1·3 in. wide. What is the area?

b A rectangle is 2·75 in. long and its area is 3·575 sq. in. What is its width?

c A rectangle is 1·3 in. wide and its area is 3·575 sq. in. What is its length?

d What are the values of $\frac{3{\cdot}575}{2{\cdot}75}$ and $3{\cdot}575 \div 1{\cdot}3$?

e How many times can you subtract 0·25 from 4·75? Is there a remainder?

f How many times can you subtract 0·5 from 4·75? Is there a remainder? If so is the remainder less than 0·5? Which arithmetical operation is this really? What fraction of 0·5 is the remainder (as a decimal)?

g Write out the first ten terms of the following set:

$$\{x \mid x = 3{\cdot}5\,n;\ n \text{ is a natural number}\}$$

h Using the results of the last question answer the following questions:

i What is the fourth term of the sequence?
ii What is $14 \div 3{\cdot}5$? What is 1·4 divided by 0·35?
iii What is 2·45 divided by 0·35?
iv What is $0{\cdot}35 \times 9$?
v Divide 2·55 by 0·35, giving the answer as a whole number with a remainder.

i Using the answers you have just obtained, complete the following:

i 1·4 divided by 0·35 =
ii $140 \div 35$ =
iii $2{\cdot}45 \div 0{\cdot}35$ =
iv $245 \div 35$ =

j By what do we have to multiply each of the first numbers in the following pairs to obtain the second?

i 0·47 ; 47
ii 0·5 ; 50
iii 0·134 ; 134
iv 2·54 ; 254
v 13·105 ; 131050
vi 20·01 ; 2001
vii 0·04 ; 400
viii 0·05 ; 5000
ix 0·002 ; 2
x 0·00001 ; 10

k If we have a fraction $\frac{a}{b}$ and we decide to multiply the numerator by a number, say 37, what do we have to do to preserve the correct value of the fraction?

l Is the following series of equalities correct? If any is wrong, say what it should be.

$$\frac{8}{12}=\frac{2}{3}=\frac{12}{18}=\frac{10}{15}=\frac{24}{30}=\frac{100}{150}$$

m If you want to divide $4{\cdot}5$ by $2{\cdot}25$, will you be able to obtain the answer by dividing 450 by 225?

n Rewrite each of the following in the form of whole numbers:

i $6{\cdot}2 \div 3{\cdot}1$
ii $7{\cdot}8 \div 0{\cdot}3$
iii $12{\cdot}88 \div 4{\cdot}6$
iv $13{\cdot}63 \div 2{\cdot}9$
v $11{\cdot}7 \div 4{\cdot}5$
vi $13{\cdot}23 \div 4{\cdot}3$
vii $11{\cdot}6 \div 3{\cdot}4$
viii $24{\cdot}6 \div 2{\cdot}9$
ix $13{\cdot}23 \div 2{\cdot}7$

o Carry out the division in i–vi of the last question.

p What is 47 divided by $0{\cdot}47$? How many times bigger than $0{\cdot}47$ is 47 ?

q What is $0{\cdot}47$ divided by 47? What fraction of 47 is $0{\cdot}47$?

r If $a < 1$ and b is a decimal fraction, $0 < b < 1$, which of the following is true?

i $\frac{a}{b} > a$
ii $\frac{a}{b} = a$
iii $\frac{a}{b} < a$

Exercise 2 **a** Explain the connexion between:

i $\frac{5}{10} \times \frac{5}{10}$
ii $0{\cdot}5 \times 0{\cdot}5$
iii $(0{\cdot}5)^2$
iv $\sqrt{0{\cdot}25}$

b Is the square root of a number greater than one
i greater or less than the number
and ii greater or less than one?

c Is the square root of a number between zero and one
i greater or less than one
and ii greater or less than one number?

d Find the squares of the following numbers:

i $0{\cdot}1$
ii $0{\cdot}4$
iii $0{\cdot}8$
iv $0{\cdot}12$
v $0{\cdot}16$
vi $0{\cdot}34$
vii $0{\cdot}56$
viii $0{\cdot}78$
ix $0{\cdot}01$
x $0{\cdot}004$
xi $0{\cdot}07$
xii $0{\cdot}0009$

e Find the square roots of the following numbers:

i 16
ii $0{\cdot}16$
iii 36
iv $0{\cdot}36$
v $0{\cdot}01$
vi $0{\cdot}0144$
vii $0{\cdot}64$
viii $0{\cdot}0004$

Exercise 3 **a** My car petrol tank holds $5\frac{1}{4}$ gallons of petrol. I intend to take it to the continent where petrol is bought by the litre. There are approximately $4\frac{1}{2}$ litres to the gallon. How many litres will fill my tank?

b Premium petrol is 10 pesetas per litre in Spain and 1·02 francs per litre in France. In Portugal it costs 6 escudos for 1 litre. Find the cost in the currency of each country of the equivalent of 4 gallons of premium petrol.

c The conversion rates of the currencies are as follows: 167·45 pesetas = 13·71 francs = 80·25 escudos = £1. Find the costs of the 4 gallons of petrol in sterling currency. (Use the prices given in 3b.)

d The car consumes 1 gallon of petrol for an average journey of 40 miles. If 1 mile is 1·609 km., find the costs of a journey involving 1000 miles in France, 2000 miles in Spain and 800 miles in Portugal and the total distance in km. Give the separate figures for each country, the total in pounds sterling and the total distance in km.

A useful way of giving a conversion table for units or currency is like the one we give here for miles and kilometres. If you want to convert 6 miles to kilometres, you find 6 in the centre column and read off the number of kilometres which is approximately equivalent in the column headed 'km.'; to convert, say, 3 km. to miles, you find again 3 in the centre column but read the equivalent in the column headed 'miles'.

km.		miles
1·609	1	0·621
3·218	2	1·242
4·827	3	1·864
6·437	4	2·485
8·046	5	3·107

Exercise 4 **a** Use the table to find the equivalent of

i 2 miles   ii 4 km.   iii 6 miles   iv 7 km

Extend the table if necessary; give all the answers correct to one decimal place.

b If 1 metre is 1·094 yards and 1 yard is 0·914 metres, construct a conversion table like the one above reading from 1 to 5 and giving results correct to two decimal places.

c From your table obtain the equivalents of 12 yards and of 14 metres, to one decimal place.

The currency of most European countries, in fact of most countries in the world is what is called a 'decimal currency'. This means that the larger coins and notes are multiples of ten of the smaller ones. This makes the use of decimals in calculation essential. Certain kinds of calculation, for instance percentages, become very easy to do. The same kinds of calculation can be done on decimals of £1 but in Great Britain we have not coins equivalent to the decimal fractions of the £. During Queen Victoria's reign a florin was introduced as £0·1 but the changes were never continued. In the countries of the European continent, coins and notes are usually only made in the following units : 1, 2, 5, 10, 25, 50, 100, 500, 1000. Some countries do not use all of these while Holland has a $2\frac{1}{2}$-gulden note and coin, Switzerland has a $\frac{1}{2}$-franc piece and Greece a 30-drachma piece. These are the only exceptions to the general rule.

Exercise 5 **a** Explain why the only denominations needed are the ones mentioned. Why for instance do they not need a 4 or a 40?

b Why does Switzerland not have a 50-centime piece?

c Do you think it would be inconvenient in Holland to have no 5-gulden note or piece?

d The Dutch currency has 100 cents to one gulden (which is about 1s. $10\frac{1}{2}$d.). They do not have a 50 cent piece nor a $\frac{1}{2}$-gulden. Do you think this would be particularly inconvenient?

e Make up an imaginary system of coinage for our own country retaining the present value of the pound sterling and using the already existing florin. Do you think there would be any advantage in keeping the half-crown? (If you use any new coins give them names different from the present ones and give their values in terms of the present coinage.)

f Construct another imaginary system based on the ten shilling unit. Say what coins you would have to dispense with and try to ensure that the prices of very small articles such as you might buy sometimes are not raised or lowered much by your system.

g All the countries which have a decimal coinage use a basic unit which lies in value between $\frac{1}{8}$d. and 2s. 3d. None has any bigger unit. Is this convenient? Could we use a system based upon the penny? In Yugoslavia the dinar is equivalent to about $\frac{1}{8}$d. sterling and the price of a car is then about 1,250,000 dinars ; in Spain the same car would cost about 100,000 pesetas. What is the cost of a £600 car in pence? Do you think it is a good idea to deal with numbers like these? Is anything in fact changed if you call, say, 100 pence a noble, and 100 nobles a royal?

Section 6 **The Venn diagram**

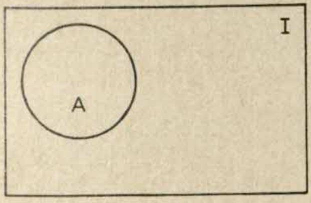
The diagrams we have been using from time to time to illustrate the way sets intersect one another are called VENN DIAGRAMS. It is important before we begin to remember that the diagram is not a set but only represents a set. It is not the only way of representing a set, in fact you have met several others in this book and the last.

Whenever we begin talking about a subject, it is a good idea to know precisely what it is we are talking about. This is especially true of sets. If we are talking about sets of people but only intend to take our sets from the set of pupils at Hotrod County School, we can exclude all other people from our discussion. We say that our UNIVERSE OF DISCOURSE is the set of pupils at Hotrod County School. In a sketch we show the universe as a rectangle and in symbols we usually call it I, U, E, or ℰ.

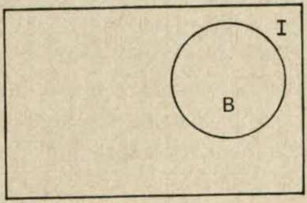
Suppose I = {pupils at Hotrod} ; and within it we have subsets:

A = {boys with spectacles}
B = {girls without spectacles)
C = {pupils with red hair}

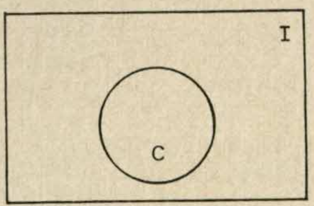
We can illustrate this in the following way:



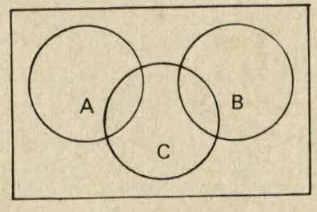
Boys with spectacles



Girls without spectacles



Pupils with red hair



All the diagrams superimposed (i.e. on top of one another)

Exercise 1 Copy the last diagram six times and shade the regions indicated. Do each on a separate diagram.

a $A \cup C$.

b $C \cup B$.

c Boys with red hair and spectacles.

d Pupils with spectacles.

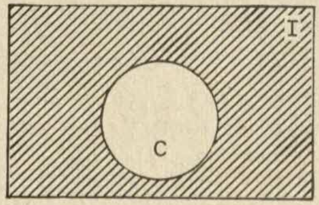
e Pupils without red hair.

f Pupils who are neither boys with spectacles nor girls with red hair and no spectacles.

Exercise 2 **a** Do you know that there are any pupils who are boys with spectacles and red hair in Hotrod School? Could the region $A \cap C$ represent a null set?

b Suppose you knew that no pupils had both spectacles and red hair. Redraw the diagram to show this correctly.

When you did Exercise 1 (e) above, you drew and shaded a diagram something like this. It is useful to have a name and a symbol for the shaded area (we already have them for the part which is not shaded). We call the shaded area the COMPLEMENT of C and use the symbol C' to refer to it. We can read it as 'not-C' or 'C prime': but we must remember that this phrase only applies within the universe we are discussing. That means that C' in Exercise 1 is the set of pupils *in Hotrod School* who do not have red hair.



Exercise 3 Using I, A, B, C as before and defining D, E, F as in this paragraph, answer the following questions.

D = {pupils who live at Moriston}
E = {pupils who cycle to school}
F = {pupils who play in a school team.}

a Describe A' and B' in words. On a diagram show A and B and shade A' and B' with different colours.

b Draw a diagram to illustrate $A \cup B$ and another to illustrate $(A \cup B)'$.

c Draw a diagram to show $A' \cap B'$.

d Do you notice anything about the two results?

e Illustrate $(A \cap B)'$ and $A' \cup B'$ on two separate diagrams.

f Can you conclude anything from (e)?

g Write down an expression which describes the redhaired pupils who cycle to school.

h Write down an expression which describes the set of children each of whom either plays in a school team or lives at Moriston.

i Write down in words a description of $(C \cap E) \cap F$ and illustrate with a Venn diagram.

j Write down in words a description of $B \cap (E \cup F)$ and illustrate with a Venn diagram.

Exercise 4 We shall take a new universe for these questions.

$I = \{\text{natural numbers}\}$
$A = \{\text{multiples of } 2\}$
$B = \{\text{multiples of } 3\}$
$C = \{\text{multiples of } 4\}$
$D = \{\text{multiples of } 5\}$
$E = \{\text{odd natural numbers}\}$
$F = \{\text{even natural numbers}\}$
$G = \{\text{prime natural numbers}\}$
$H = \{\text{Fibonacci numbers}\}$

a Are two of the subsets of I identical?

b Describe in words E' and F'.

c Complete the following with one symbol in each case:

i $E' =$ ii $F' =$ iii $E \cap F =$ iv $C \cup A$

d What do you call $B \cap D$?

e Write out in set notation displaying the first six numbers in each case where this is possible:

i $D \cap G$
ii $B \cap E$
iii $A \cap H$
iv $G \cap H$
v H'
vi C'
vii $E \cup F$
viii $E \cap F$
ix $(E \cup F)'$
x $E' \cap F'$
xi $(A \cap B)'$
xii $A \cup B'$

Exercise 5 Can you illustrate with Venn diagrams the rules that $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ whatever the universe of discourse is? (These two laws are very important ones and are called the De Morgan laws; see if you can find out who De Morgan was and when he lived.)

Exercise 6 In a sixth form of a school, it so happens that a group who share a form-room study the following subjects: French, English, German, physics and mathematics. It is a rule in the school that sixth-formers can only take German if they also take French; pupils are allowed to take English with another language or with mathematics but not with physics. There is also a rule that pupils taking physics must also take mathematics unless they have already passed the examination. Some physics students have done so. Every pupil must take two subjects and not more than two.

In the answers to the following, use the letters I, F, E, G, P, M to symbolize the sets of pupils taking the subjects.

a Illustrate with a Venn diagram the pupils taking German and those taking French.

b Add to your last sketch those taking English.

c How many are in the set $E \cap G$?

d Are there any pupils doing mathematics who are not also doing physics? If so, what other subject may they be taking?

e What other subjects must the pupils taking physics but not mathematics have chosen?

f Illustrate the whole set on a single Venn diagram. You will need five closed curves and a rectangle. Be sure that all the possible arrangements of subjects are shown on the diagram.

If we take I, the universe of discourse, as the set of natural numbers, as we did in Exercise 4 and two sub-sets which do not overlap, like sets E and F in the same exercise, we have a rather special situation. Usually subsets are not like E and F: you will remember that $E \cup F = I$ and $E \cap F = \phi$. In other words, the two subsets do not overlap, and together they make up the whole universe. Sets which do not overlap are said to be DISJOINT sets. We are going to meet again in the next section, disjoint sets which together make up a universe.

Section 7 **Powers again**

Note that *power notation* is often called *index notation* and the small numbers representing the powers are often called *indices.*

Exercise 1 **a** Write each of the following in a shorter form:

i $a \times a \times a$
ii $f \times f \times f \times f$
iii $k \times k$
iv $2 \times n \times n$
v $8 \times g \times g \times g \times g$
vi $5 \times u$
vii $6 \times 7 \times j \times j \times j \times j$
viii $5 \times 8 \times m^2 \times m^7$
ix $9 \times 5 \times 4 \times s^6 \times s^8 \times s^2$
x $8 \times d \times d^6$
xi $3 \times t^4 \times 7 \times t^8 \times 9 \times t^9$
xii $y^8 \div y^6$
xiii $r^7 \div r^3$
xiv $p^5 \div p$
xv $h^9 \div h^8$
xvi $v^6 \div v^5$
xvii $\dfrac{3h^7}{h^3}$
xviii $\dfrac{9k^8}{k^6}$
xix $\dfrac{10r^7}{5r^4}$
xx $\dfrac{16x^{10}}{4x^7}$
xxi $\dfrac{27d^{17}}{9d}$

b Write each of the following in a fuller form:

i b^2
ii k^3
iii $4z^4$
iv $17h^3$
v $23f^5$
vi $4p^2 \times p^4$

c Work out as a number:

i 2^5
ii 2^6
iii 2^7
iv 2^8
v 2^9
vi 2^{10}
vii $2^3 \times 2^6$
viii $2^2 \times 2^7$
ix $2^6 \times 2^8$
x $2^9 \times 2^5$
xi $2^5 \times 2^5$
xii $(2^5)^2$
xiii $2^3 \times 2^3 \times 2^3$
xiv $(2^3)^3$
xv $2^2 \times 2^2 \times 2^2 \times 2^2$
xvi $(2^2)^4$
xvii $(2^2)^5$
xviii $(2^3)^2$
xix $(2^2)^3$
xx $(2^3)^4 \times (2^4)^3$

d Express in the simplest possible form:

i $3^5 \div 3^3$
ii $4^5 \div 4^3$
iii $4^7 \div 4^6$
iv $7^6 \div 7^4$
v $a^6 \div a^3$
vi $g^{17} \div g^{15}$
vii $k^9 \div k^8$
viii $h^9 \div h^4$

e i State in words the rules you use for multiplication and division of expressions in index form.

ii Does the rule apply to expressions like:

$$4^5 \times 3^5 \text{ or } 5^4 \times 4^5 \text{ or } a^8 \div b^5 \text{ or } d^6 \div f^3?$$

If the rules do apply, then give the simplified forms of these expressions.

iii Can you give a rule for working out $(a^p)^q$?

f Write in an alternative form: 2^1, 3^1, 4^1, 5, 6, 7.

g Check the following working and say whether or not it is correct.

i $$\frac{g^4 \times g^6}{g^7} = \frac{{}^1\cancel{g} \times {}^1\cancel{g} \times {}^1\cancel{g} \times {}^1\cancel{g} \times {}^1\cancel{g} \times {}^1\cancel{g} \times {}^1\cancel{g} \times g \times g \times g}{{}_1\cancel{g} \times {}_1\cancel{g} \times {}_1\cancel{g} \times {}_1\cancel{g} \times {}_1\cancel{g} \times {}_1\cancel{g} \times {}_1\cancel{g}} = g^3$$

ii $$\frac{g^4 \times g^6}{g^7} = g^{4+6-7} = g^3$$

h Now read and check the following pairs of statements:

i $j^4 \div j^4 = j^0$; $\quad j^4 \div j^4 = \dfrac{{}^1\cancel{j} \times {}^1\cancel{j} \times {}^1\cancel{j} \times {}^1\cancel{j}}{{}^1\cancel{j} \times {}^1\cancel{j} \times {}^1\cancel{j} \times {}^1\cancel{j}} = 1$

ii $p^3 \div p^3 = p^0$; $\quad p^3 \div p^3 = \dfrac{{}^1\cancel{p} \times {}^1\cancel{p} \times {}^1\cancel{p}}{{}^1\cancel{p} \times {}^1\cancel{p} \times {}^1\cancel{p}} = 1$

iii $2^5 \div 2^5 = 2^0$; $\quad 2^5 \div 2^5 = \dfrac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = 1$

iv $10^5 \div 10^5 = 10^0$; $\quad 10^5 \div 10^5 = \dfrac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10 \times 10 \times 10} = 1$

v $x^a \div x^a = x^0$; $\quad x^0 = 1$

We have been looking at multiplication and division of various types of expressions in index form. You now should remember certain results:

i how to multiply
ii how to divide
iii when you cannot do either (i) or (ii)
iv any number raised to power 0 is equal to 1

There is another question we may want to answer now:
What do we put into the frames to satisfy the following:

$\square \times \square = x^6$? This can also be expressed in another way as $\sqrt{x^6} = \square$; $\sqrt{\ }$ is read *the square root of.*

Exercise 2

a $x^3 \times x^3 = x^{\square}$; $\quad x^{\square} \times x^{\square} = x^6$

b $p^5 \times p^5 = p^{\square}$; $\quad p^{\triangle} \times p^{\triangle} = p^{10}$

c $x^1 \times x^1 = x^{\triangle}$; $\quad x^{\triangledown} \times x^{\triangledown} = x^2$

d $x^9 \times x^9 = x^{\triangle}$; $x^{\triangle} \times x^{\triangle} = x^{18}$

Exercise 3 **a** What does the sign $\sqrt{}$ mean? Explain.

b Write down the natural numbers equal to the following:

i $\sqrt{4}$
ii $\sqrt{16}$
iii $\sqrt{49}$
iv $\sqrt{81}$
v $\sqrt{100}$
vi $\sqrt{196}$

c Find the numbers which fill the frames to make true statements of the following:

i $25 \times 25 = \Box$; $\sqrt{625} = \triangle$.
ii $36 \times 36 = \nabla$; $\sqrt{1296} = \Box$.
iii $4{\cdot}9 \times 4{\cdot}9 = \Box$; $\sqrt{24{\cdot}01} = \nabla$.

d Fill the following frames with natural numbers to make true statements:

i $\sqrt{3^2} = \Box$
ii $\sqrt{\Box^2} = 5$
iii $\sqrt{\Box^2} = 7$
iv $(3^2) = \Box$
v $\sqrt{(7^2)^2} = \Box$
vi $\sqrt{(\Box^2)^2} = 81$
vii $\sqrt{x^2} = \Box$
viii $\sqrt{x^{18}} = \Box$
ix $\sqrt{x^8} = \Box$
x $\sqrt{\nabla^{10}} = x^5$
xi $\sqrt{\nabla^6} = x^3$
xii $\sqrt{x^{\Box}} = x^8$

e Simplify the following expressions:

i $(2^2)^2$
ii $(3^1)^2$
iii $(x^3)^2$
iv $(x^2)^3$
v $\sqrt{(4^2)^2}$
vi $\sqrt{(2^2)^2}$
vii $\sqrt{(2^1)^4}$
viii $\sqrt{(2^2)^3}$

f If $x^3 = 27$, $x = 3$ (provided x is a natural number) and x is then said to be the *cube root* of 27. Find the cube roots of the following numbers. (Give only natural numbers for your results.)

i 8
ii 64
iii 343
iv 1
v 1728
vi 0

g i If a box in the shape of a cube contains 8 cu. yds. of sand, how long is the box?

ii If a large cube is made of 64 smaller cubes each 1 in. long, how high is the larger cube?

iii How many cubic feet is 1728 cu. in.?

h The sign for a cube root is $\sqrt[3]{}$. Find the values of the following.

i $\sqrt[3]{1000}$
ii $\sqrt[3]{x^3}$
iii $\sqrt[3]{x^6}$

i Check the following statements and say whether they are true or false:

i $\sqrt[3]{9} = 3$
ii $\sqrt[3]{27} = 9$
iii $\sqrt[3]{y^3} = y$
iv $j^2 \times j^2 \times j^2 = j^6$
v $k^4 \times k^4 \times k^4 = k^{12}$
vi $\sqrt[3]{c^{12}} = c^4$
vii $\sqrt[3]{h^9} = h^3$
viii $\sqrt[3]{p^{27}} = p^3$

Scientists often need to compare numbers but they find it hard to do unless the numbers are all expressed in the same way. Differences between numbers can be very misleading too. 10·241 is obviously about nine times as big as 1·132, although their difference is only about 9. 1,675,428 is not so very different from 1,677,536 although the difference of the numbers is 2,108. One way of getting a comparison is by using percentages. You could find what percentage increase will make 1·132 into 10·241 (10·241 is about 900% of 1·132) and compare it with the corresponding increase on the other number (1,677,536 is about 100·12% of 1,675,428). This method unfortunately requires a good deal of calculation and the accuracy is not always useful. Scientists often use a form of expression called 'standard form'. This always gives a number as a unit with decimal fractions together with a power of ten. For instance,

$10{\cdot}241 = 1{\cdot}0241 \times 10^1$ said to be of order 1
$1{,}675{,}428 = 1{\cdot}675428 \times 10^6$ order 6
$286 = 2{\cdot}86 \times 10^2$ order 2
$3{,}497 = 3{\cdot}497 \times 10^3$ order 3
$465{\cdot}87 = 4{\cdot}6587 \times 10^2$ order 2

This gives a rough means of comparison between large numbers. The second number in the list can be seen to be about 10^5 or a hundred thousand times the first while it is only 10^4 times the last number.

Exercise 4 **a** The moon is about 238,860 miles away from the earth. Express this distance in miles in standard form. The sun is about 93 million miles from the earth. Express this distance in standard form. About how many times greater is the distance of the sun?

b The distances of the planets from the sun are given approximately in the following table. Convert each to standard form.

Planet	Mean distance from sun
Mercury	36 million miles
Venus	67·2 million miles
Earth	92,900,000 miles
Mars	141,600 thousand miles
Jupiter	$483{\cdot}4 \times 10^6$ miles
Saturn	$88{\cdot}63 \times 10^7$ miles
Uranus	1,782·4 million miles
Neptune	$27{\cdot}929 \times 10^8$ miles
Pluto	$0{\cdot}36719 \times 10^{10}$ miles

About how many times farther out from the sun is Pluto than (i) the earth, (ii) the planet nearest to the sun?

We have talked before about finding the factors of numbers. Any set of numbers which multiply together to give a number are a set of factors of that number. Putting it the other way round, we can say that if a number, x, divides by another number, y, then y is a factor of x.

Exercise 5 **a** Find six pairs of factors which multiply to give 60.

b Find four numbers other than 1 which multiply to give 280.

c If the factors are prime numbers, they are described as 'prime factors'. Find the set of prime factors of 105.

d Which number is a factor of any number?

e What are the factors of prime numbers?

f What are the factors of 47?

Index notation gives a convenient way of writing factors down in certain cases. For instance,

$$\begin{aligned} 17{,}640 &= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 7 \\ &= 2^3 \times 3^2 \times 5 \times 7^2 \end{aligned}$$

When we want to find the factors of a number we can do it step by step. For example:

$$\begin{aligned} 3{,}600 &= 2 \times 1{,}800 \\ &= 2^2 \times 900 \\ &= 2^3 \times 450 \\ &= 2^4 \times 225 \\ &= 2^4 \times 3 \times 75 \\ &= 2^4 \times 3^2 \times 25 \\ &= 2^4 \times 3^2 \times 5^2 \end{aligned}$$

Exercise 6 Factor the numbers following:

a 2500
b 2401
c 1296
d 25,600
e 187
f 221
g 1728
h 5280
i 35,840
j 2491
k 169
l 2197

Can you identify the numbers in parts g and h as numbers you have met before?

When we have factorized numbers into their prime factors and expressed them in index notation, finding the square roots of those numbers which have whole-number square roots is very simple. We can see that

$\sqrt{16} = \sqrt{2^4} = 2^2$; $\sqrt{400} = \sqrt{2^4 \times 5^2} = 2^2 \times 5$;

$\sqrt{3600} = \sqrt{(2^4 \times 3^2 \times 5^2)} = 2^2 \times 3 \times 5$.

If we do not want the square roots in factor form it is a simple matter to multiply them up.

Exercise 7 Find the square roots of the numbers in Exercise 6. a, b, c, and d by this method. Multiply the factors up. Square your answers by ordinary arithmetic to check them. If they are wrong, find out where you have made your mistakes.

Reinforcement Exercise 1 1 Find the sum of the interior and the sum of the exterior angles of each of the following:

i a triangle
ii any quadrilateral
iii any pentagon
iv any hexagon

2 Find the size of each interior and exterior angle of the following:

i a regular pentagon
ii a regular hexagon
iii a regular heptagon
iv a regular octagon
v a regular enneagon
vi a regular dodecagon

Find a method of checking your results and then check them.

Section 8 **Sets in algebra**

Exercise 1 **a** $I = \{2a, 3a^2, 4a, 4a^2, 7a^2, 5a, 8a\}$
$A = \{2a, 4a, 5a, 8a\}$
$B = \{3a^2, 4a^2, 7a^2\}$

i Is $A \cup B = I$?

ii $A \cap B = \phi$?

iii What do you notice about the way the two subsets A and B have been chosen? What have the terms in each set in common?

b By the same means, find two disjoint subsets of the following:
$I = \{2b, 3b^3, 9b^3, 6b^3, 7b, b, 2b^3\}$

c Find three disjoint subsets of the following:
$I = \{4x, 4y, 7x, 8x, 7y, 7y^2, 8y^2, 9y, y^2\}$

d Find four disjoint subsets of the following:
$I = \{3p, 5p^3, 2p, 6p^2, 7, 5p^2, 5p, 9p^3, 6p^3, 3, 11p^2\}$

If you have followed the principle we used in the first question, you have been choosing for the subsets terms which have a common property. Check your answers carefully in order to be sure how these subsets were chosen. Now go on to the next exercise.

Exercise 2 Form subsets of each of the following sets according to the rules you learnt in the last exercise.

a $I = \{4x^2, 5x, 6, 9x^2, 8x, 1\}$

b $I = \{7x^3, 6x^2, 5x, 7, 8x^3, 5x^2, 6x, 4\}$

c $I = \{14x^4, 7x^2, 8, x^4, 3x^3, 7x, 9, 2x^4, 1\}$

d $I = \{8, 4x^2, 9x^3, \frac{1}{2}x^4, \frac{3}{4}x^5, 9x^2, 6x^3, 6x^4, x^5\}$

e $I = \{7, 5x, 6y, 7z, 9z, 3x, 7x, 4\}$

The subsets you have been finding in each of the questions above each contain terms which are similar in one respect. Can you now express the similarity in words? The terms of each subset are like one another, in fact we say they are 'like terms'. They can be added and subtracted from each other, whereas unlike terms cannot. You have met this idea before in Book 1. When you want to simplify an expression, the first job is to *think* of its subsets of like terms and simplify each of these. Here is an example: we are using the example of Exercise 1a.

$$\begin{aligned} 2a+3a^2+4a+4a^2+7a^2+5a+8a &= 2a+4a+5a+8a+3a^2+4a^2+7a^2 \\ &= 19a+14a^2 \end{aligned}$$

You will note that the last two terms left are not like terms: we cannot simplify any further.

Exercise 3 Simplify each of the following; they are based on the previous exercises.

a $2b+3b^3+9b^3+6b^3+7b+b+2b^3$

b $4x+4y+7x+8x+7y+7y^2+8y^2+9y+y^2$

c $3p+5p^3+2p+6p^2+7+5p^2+5p+9p^3+6p^3+3+11p^2$

d $4x^2+5x+6+9x^2+8x+1$

e $7x^3+6x^2+5x+7+8x^3+5x^2+6x+4$

f $14x^4+7x^2+8+x^4+3x^3+7x+9+2x^4+1$

g $8+4x^2+9x^3+\frac{1}{2}x^4+\frac{3}{4}x^5+9x^2+6x^3+6x^4+x^5$

h $7+5x+6y+7z+9z+5y+7x+4$

You will need often to add up algebraic terms in the future. This technique is a very useful one to remember.

Section 9 **About integers**

We can express whole numbers as differences of natural numbers. For instance $(6-4)$ can represent 2 and $(9-3)$ can represent 6.

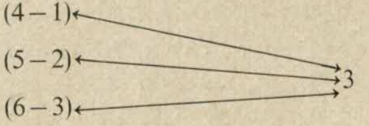
Exercise 1 **a** What are the numbers represented by $(12-4)$, $(6-3)$, $(2-0)$?

b Write down a difference to represent the number 7. Is there only one difference you could use?

c What can you say about the following numbers?

$$(5-3), (3-1), (24-22), (102-100)$$

d



etc.

What sort of mapping does this represent?

There are many differences which represent one number. We are going to use the following symbol $(4 \sim 1)$ to mean the whole family of all those differences which are 'the same' as $(4-1)$; the family will include $(7-4)$, $(24-21)$, $(9-6)$, $(3-0)$ and many others. Any one of these will do to represent the family; we shall often use the simplest and write the family $(3 \sim 0)$.

Exercise 2 **a** Write down four members of the family $(6 \sim 4)$.

b Which of the following are of the same family?

i $(9 \sim 5)$
ii $(5 \sim 4)$
iii $(12 \sim 11)$
iv $(18 \sim 11)$
v $(54 \sim 53)$
vi $(76 \sim 72)$

c Write down the simple form of the family which represents 5.

These families, each of which is an *integer*, are numbers and behave like 'ordinary' numbers. We can add and multiply them as we can natural numbers. In each case the result is an integer.

We can add them like this:

Add $(5 \sim 3)$ to $(7 \sim 4)$; the result is $(5+7) \sim (3+4)$ or $(12 \sim 7)$.

Exercise 3 **a** Add $(5 \sim 2)$ to $(6 \sim 4)$

b $(9 \sim 3)+(6 \sim 5)$

c $(14 \sim 5)+(5 \sim 0)$

d $(3 \sim 0)+(6 \sim 0)$

e $(3 \sim 9)+(6 \sim 5)$

f $(0 \sim 2)+(0 \sim 3)$

g $(0 \sim 4)+(0 \sim 5)$

h $(a \sim 4)+(b \sim 2)$

i $(4 \sim a)+(2 \sim b)$

j $(w \sim x)+(y \sim z)$

In the last three examples, you have been using symbols for numbers. It is always very important that we should say what our symbols mean, otherwise no one can ever be expected to be able to understand. In this case we must state that a, b, w, x, y, z are variables which can take any value which is a natural number.

Compare $(9 \sim 0)$ and $(0 \sim 9)$. The first represents the integer which we often write as 9; the second does not represent any kind of number we have met before. Let us find out more about these new numbers.

Exercise 4 **a** Add $(9 \sim 0)$ to $(0 \sim 9)$.

b Add $(0 \sim 9)$ to $(9 \sim 0)$.

c What is the simplest form of $(9 \sim 9)$?

d What integer is represented by $(0 \sim 0)$?

e $(4 \sim 0)+(0 \sim 4)$. Rewrite in the simplest form.

f $(5 \sim 0)+(0 \sim 5)$. Rewrite in the simplest form.

$(5 \sim 0)$ represents the number we often write as 5 but which as you will see, we should write as $^{+}5$; $(0 \sim 5)$ represents a new number which we shall represent as $^{-}5$. The two numbers are read out as 'plus five' or 'pos 5' and 'minus five' or 'neg 5'. We shall call the first a 'positive integer' and the second a 'negative integer'.

We already know something about these negative and positive integers; we began to learn their arithmetic at the beginning of this section. For instance we know that if we add two positive integers we get another positive integer. $(5 \sim 0)+(3 \sim 0) = (8 \sim 0)$ for instance. Or, putting that into the other notation, $^{+}5+{}^{+}3 = {}^{+}8$.

See if you can say what are the other things we have discovered. The next exercise will guide you, step by step.

Exercise 5 **a** What do we get when we add two negative integers? Try two of your own to find out. They all behave the same way.

b What do we get when we add a positive integer to its *corresponding* negative (i.e., $^{+}5$ and $^{-}5$)?

c Do we always get the same kind of result when we add a positive and a negative integer? Try $(5 \sim 0)$ and $(0 \sim 3)$ and then $(3 \sim 0)+(0 \sim 5)$.

d Does it matter which way round we put the numbers we are going to add? For instance is ${}^{-}3+{}^{-}5$ the same as ${}^{-}5+{}^{-}3$? or ${}^{+}5+{}^{-}7$ the same as ${}^{-}7+{}^{+}5$?

Exercise 6 Find the solution sets of the following:

a $\{x \mid x+4 = 8;\ x \text{ is a natural number}\}$

b $\{x \mid x+4 = 0;\ x \text{ is a natural number}\}$

c $\{x \mid x+{}^{+}4 = 0;\ x \text{ is an integer}\}$

There are some questions which have no answer in natural numbers; we have met some of them before (e.g., the solution set of $\{x|3x = 5;\ x \text{ is a natural number}\}$ is $\{\ \}$) and we needed some special numbers for the answers. These numbers were fractions or ratios of natural numbers and are called '*ratio*nal numbers'. Now we have found another sort of question which cannot be answered with natural numbers. In this case the special numbers we need to use are the integers. In the questions which follow, you will have to consider which system of numbers you are to use before you answer the question. In some cases there may not be any solution, that is the solution set will be $\{\ \}$.

Exercise 7 Find the solution sets of the following:

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a $\{x \mid x+7 = 12;\ x \text{ is a natural number}\}$

b $\{x \mid x+{}^{+}7 = {}^{+}12;\ x \text{ is an integer}\}$

c $\{y \mid y+{}^{+}19 = {}^{+}8;\ y \text{ is an integer}\}$

d $\{z \mid z+{}^{+}53 = {}^{+}27;\ z \text{ is an integer}\}$

e $\{a \mid a-16 = 4;\ a \text{ is a natural number}\}$

f $\{b \mid b+{}^{-}16 = {}^{+}4;\ b \text{ is an integer}\}$

g $\{p \mid p+{}^{-}6 = 0;\ p \text{ is an integer}\}$

h $\{q \mid q-25 = 0;\ q \text{ is a natural number}\}$

i $\{f \mid 3f = 11;\ f \text{ is an integer}\}$

j $\{j \mid 4j = 16;\ j \text{ is an integer}\}$

k $\{k \mid 6k = 14;\ k \text{ is a rational number}\}$

l $\{g \mid 27g = 54;\ g \text{ is a natural number}\}$

If you have been thinking carefully about these questions, you will have noticed something strange about the way we have written the numbers in some of the equations. Can you see what it is? We shall return to this point later.

Exercise 8 Find the solution sets of the following:

a $\{x \mid x+5 = 8;\ x \text{ is a natural number}\}$

b $\{x \mid x+{}^{+}5 = {}^{+}8;\ x \text{ is an integer}\}$

c When you solve $x+5 = 8$ or $x+{}^{+}5 = {}^{+}8$ what has this to do with subtraction?

d $\{x \mid {}^{+}5+{}^{+}3+x = {}^{+}5;\ x \text{ is a negative integer}\}$

e $\{x \mid {}^{+}8+x = {}^{+}5;\ x \text{ is a negative integer}\}$

f $\{x \mid {}^{+}12+x = {}^{+}8;\ x \text{ is a negative integer}\}$

Let us now look again at positive and negative integers to decide how to add them easily. $(6 \sim 0)+(0 \sim 4) = (6 \sim 4)$ using the rule we made at the beginning. $(6 \sim 4) = (2 \sim 0)$. Why? If we use the other notation we get ${}^{+}6+{}^{-}4 = {}^{+}2$. Let us try another. $(2 \sim 0)+(0 \sim 9) = (2 \sim 9) = (0 \sim 7)$ or in the simple notation: ${}^{+}2+{}^{-}9 = {}^{-}7$.

Exercise 9 In the following exercise, work through the method we have just used and rewrite the answer in the simple notation in each case.

a ${}^{+}12+{}^{-}4$

b ${}^{+}5+{}^{-}2$

c ${}^{+}4+{}^{-}1$

d ${}^{+}10+{}^{-}5$

e ${}^{+}2+{}^{-}2$

f ${}^{+}2+{}^{-}1$

g ${}^{+}5+{}^{-}8$

h ${}^{+}56+{}^{-}59$

i ${}^{-}7+{}^{-}5$

j ${}^{-}7+{}^{+}5$

k ${}^{+}7+{}^{-}5$

l ${}^{+}7+{}^{+}5$

Examine the numbers you were working with and the answers very carefully to see if you can see a method of getting the answer quickly without going through the proof as you have been doing.

When you solved the equation $x+5 = 8$ or $x+{}^{+}5 = {}^{+}8$ did you think '$x = 8-5$' in the first case or '$x = {}^{+}8-{}^{+}5$' in the second? This is what we mean by subtraction. For example, if we ask 'What number added to 5 gives 8?' the answer is '$8-5$'. If we ask the same question with integers both positive and negative we shall be able to find out how to subtract these numbers easily. Try $x+{}^{+}8 = {}^{+}5$, $x = {}^{+}5-{}^{+}8$. But we already know that the solution of $x+{}^{+}8 = {}^{+}5$ is $x = {}^{-}3$. So it follows that ${}^{+}5-{}^{+}8 = {}^{-}3$.

Exercise 10 Simplify the following:

a $^{+}4-{}^{+}5$
b $^{+}7-{}^{+}9$
c $^{+}12-{}^{+}14$
d $^{+}4-{}^{+}7$
e $^{+}2-{}^{+}8$
f $^{+}3-{}^{+}12$
g $^{+}16-{}^{+}13$
h $^{+}54-{}^{+}27$
i $^{+}27-{}^{+}54$
j $^{-}7-{}^{+}5$
k $^{-}5-{}^{+}7$
l $^{-}32-{}^{+}43$
m $^{-}4-{}^{+}5$
n $^{-}7-{}^{+}9$
o $^{-}12-{}^{+}14$
p $^{+}5+{}^{+}7$
q $^{+}7-{}^{+}5$
r $^{-}5-{}^{+}7$
s $^{+}7+{}^{+}5$
t $^{-}5+{}^{+}7$
u $^{+}5-{}^{+}7$
v $^{-}7-{}^{+}5$
w $^{-}5+{}^{-}7$
x $^{+}5+{}^{-}7$

If you look at the last few questions you will see that there is a large number of fives and sevens in various combinations and all the problems are different. But there are some possibilities still missing. Can you see which they are? If you examine the whole exercise, you will probably notice something. These problems will be dealt with in a later section: we have not finished with integers yet.

Reinforcement Exercise 2 Simplify each of the following:

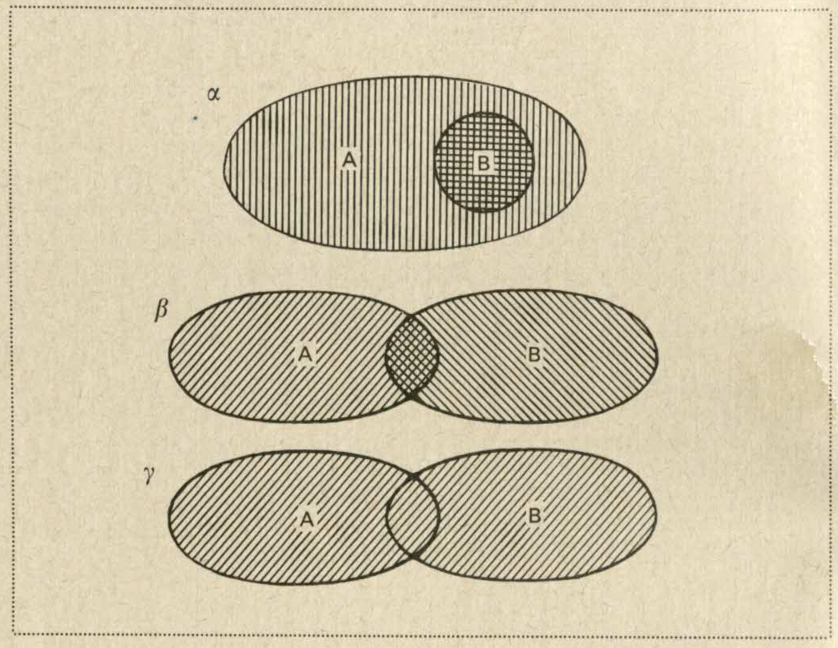
i $3x+4x^2+5x^3-2x^2-x+3$
ii $5y+4-2y^2+3y^3-2y^2-3y^3-y$
iii $4a-3b+2a-4a+5c-4a+b$
iv $2x^2-3xy+4x^2-y^2-3y^3-y^2+2xy$
v $a+b-ab-b^2+a^2-a+b^2-ab-b$
vi $3x\times 4y$

Expand:
vii $3(x+y)$
viii $x(2+y)$
ix $4a(a+b)$
x $2b(a+3c)$

Answers on page 233.

Section 10 **Some further questions on sets**

Exercise 1 **a**



Which of these diagrams, α, β, γ, are intended to illustrate

i $A \cap B$ ii $A \cup B$ iii $B \subset A$?

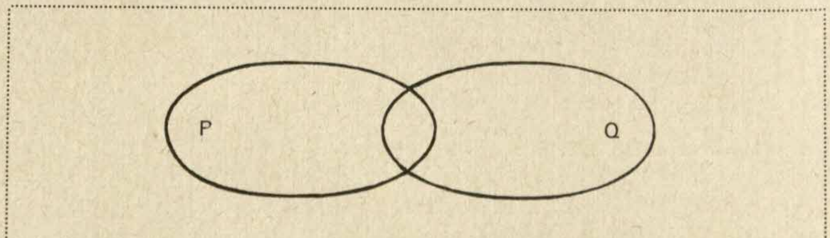
b Suppose set A = {♭, ♯, ♮} and set B = {♯, x}. Say whether the following are true or false:

i $A \cup B$ = {♭, ♯, ♮, ♯, x}

ii $A \cap B$ = {♯}

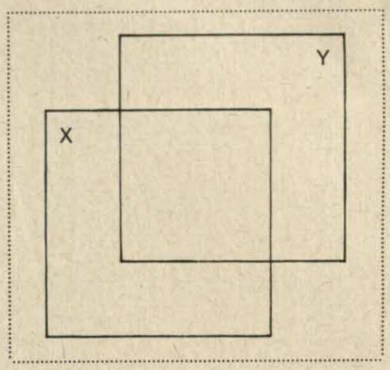
iii $B \subset A$

c



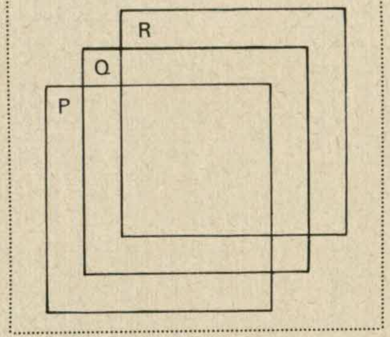
Copy the diagram twice and illustrate (i) $P \cup Q$ and (ii) $P \cap Q$.

d Copy the diagram twice and illustrate (i) X ∪ Y and (ii) X ∩ Y.



e Copy the diagram four times and illustrate

i P ∩ Q
ii (P ∩ Q) ∩ R
iii (P ∩ Q) ∪ R
iv R ∩ P
v Is (R ∩ P) ⊂ Q?



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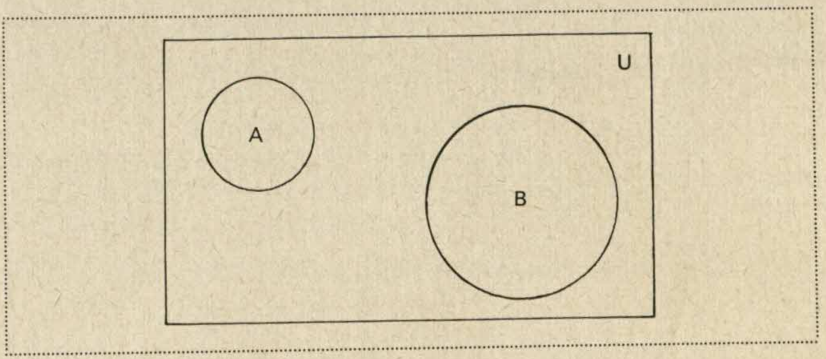


U is the universe of discourse. Copy the diagram six times and shade in the following:

i A′
ii B′
iii (A ∪ B)′
iv B′ ∩ A
v A ∪ B
vi A′ ∩ B′

By looking at the diagrams you have just drawn, decide which of the following statements are true and which false.

i $A' \subset U$ iii $B' \cap A = A$ v $(A \cup B)' = A' \cap B'$
ii $B' \subset A$ iv $(A \cup B) \subset U$

If $U = \{a, b, c\}$ then the possible subsets are
$A = \{a, b\}$ $C = \{a, c\}$ $E = \{b\}$ $G = \{\ \} = \phi$
$B = \{b, c\}$ $D = \{a\}$ $F = \{c\}$
$U = \{a, b, c\}$ is usually also regarded as a subset of itself. Subsets A to G are called *proper subsets* of U.

Exercise 2 **a** i Make a list of all the subsets of $U = \{p, q\}$. How many subsets are there?

ii Repeat this for the set $U = \{j, k, l, m\}$.

b Examine this pattern of numbers. Find out how it is constructed and explain what it has to do with the subsets of a set. Does the pattern work with 5, 6 and more elements?

1 1	one element
1 2 1	two elements
1 3 3 1	three elements
1 4 6 4 1	four elements

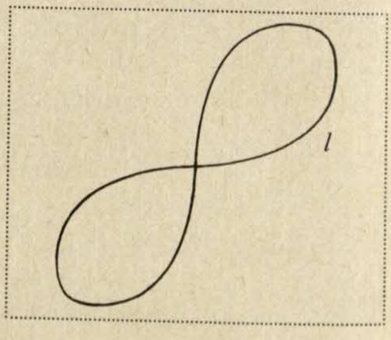
. . . . . and so on . . . . .

This pattern of numbers is called Pascal's triangle.

c If $N = \{\text{all natural numbers}\}$
and $A = \{\text{all odd numbers}\}$
and $B = \{\text{all even numbers}\}$, name the following sets:
i A' iii $A \cap B$ v $N \cap A$
ii B' iv $A \cup B$ vi $(A \cup B)'$

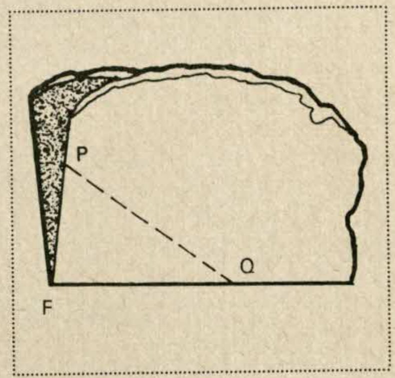
d If $I = \{\text{points in a whole plane}\}$
and $P = \{\text{points in a straight line in the plane}\}$
and $Q = \{\text{points on the circumference of a circle in the plane}\}$,
describe the possible solution sets of $\{R | P \cap Q = R\}$.

e If in addition to the data of the previous question, $S = \{\text{points on a line, } l\text{, illustrated, lying in the plane}\}$ describe the possible solution sets of (i) $\{T | P \cap S = T\}$ and (ii) $\{V | Q \cap S = V\}$.



Section 11 **More about paperfolding and geometry**

Exercise 1 For this exercise, fold four pieces of paper as in the diagram and make the creases sharp.



a How many thicknesses of paper are there?

b If you were to cut along PQ through all the thicknesses, how many edges of length PQ would there be to the piece cut off?

c Before you cut and open out the paper, say what sort of shape will be produced. Will it for instance have 3, 4, 6, or 8 sides. Will they be equal or unequal? Do you recognize the figure? Now cut and open out the paper. Were you right?

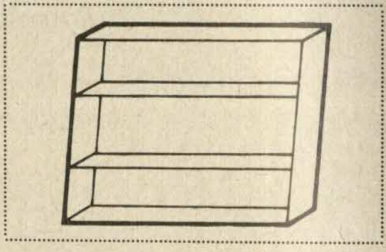
d Cut each of your folded pieces of paper, in one case making PF = FQ and in the others make PF and FQ different sizes. Are all of your figures rhombuses?

e Do you think the square is merely a special sort of rhombus? If so, what is special about it?

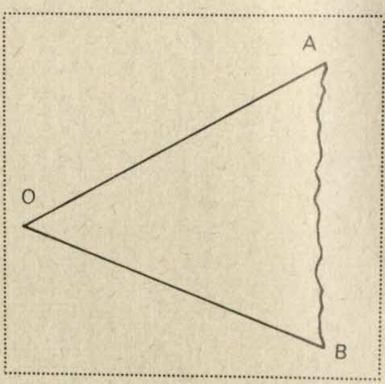
f Are the diagonals of a rhombus equal? Are the diagonals of a square equal?

Exercise 2 **a** Take four equal strips: card will do; Meccano will be better. Hinge them together at the ends to make a rhombus. Can they be moved to form a square?

b A handyman has made this bookcase which should be square, but it leans. Consider what you know about rhombuses and squares and say how he could have made certain that it was square before putting the back on it.



Exercise 3 **a** Cut a single piece of paper as in the sketch. Can you see how to fold it so that the fold bisects angle AOB?



b Can you do the same with a 'square' corner of a sheet of paper? How big are the angles so formed?

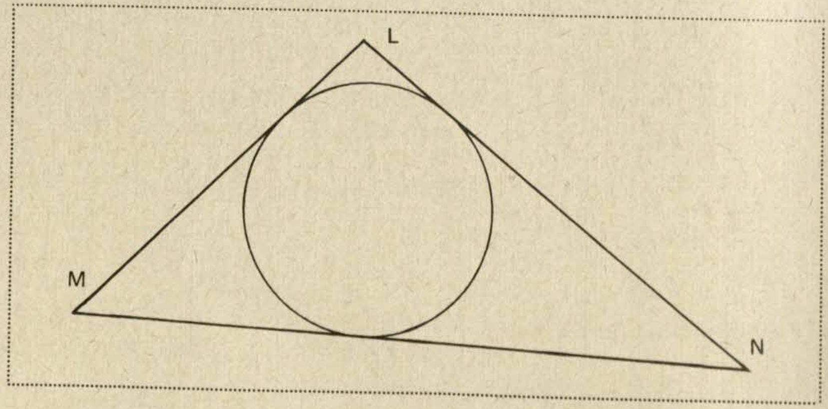
c Cut out an acute angled triangle with unequal sides (a scalene triangle) and fold it to bisect all the angles. What do you notice?

d Do the same with

i an isosceles triangle
ii an equilateral triangle
iii a right angled triangle
iv an obtuse angled triangle.

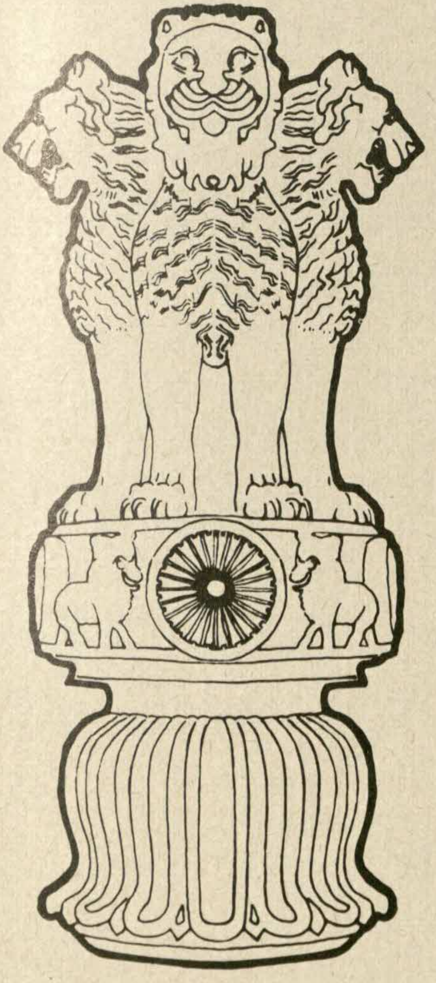
Do you notice some rather special results from some of the triangles?

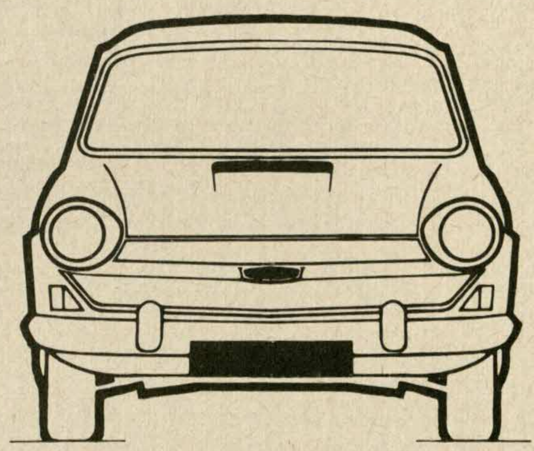
e

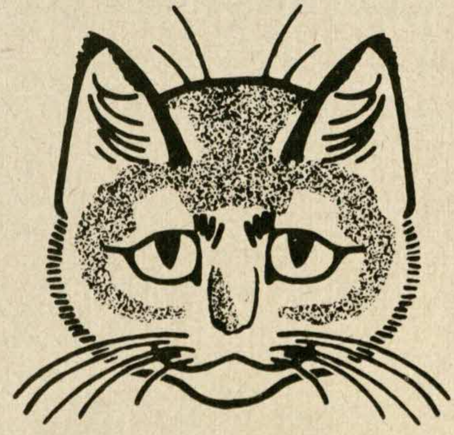


Do this very carefully indeed. Draw a circle of about 2 in. on a sheet of plain paper. Draw a triangle LMN whose sides just touch the circle. Now cut out the triangle and fold to bisect the angles. What do you notice about the centre of the circle? Will this work with other triangles? Try it with several different ones and find out.

If we fold a piece of paper once and cut through it from the fold or prick with a pin through the double thickness of the paper, when we







open the paper out we find the same arrangement of points and lines on either side of the fold. If we place a mirror along the fold, the paper we can see together with the reflection in the mirror form the same picture as the whole paper. The picture is said to be *symmetrical* and the two sides are said to be *reflections* of one another. The crease is the *axis of symmetry*.

Exercise 4 **a**

A B C D G H

Many of the letters of the alphabet are symmetrical in the way just described. A is; R is not. Imagine all the letters printed as the illustrations suggest and say which are symmetrical and which are not. Draw in the axis of symmetry of each. Does any letter have more than one axis of symmetry?

b Do the same for the ordinary English numerals.

Section 12 **Integers again**

When we last looked at integers, we learnt how to add and subtract positive integers and how to add negative integers. We did not discover how to subtract negative integers. In this section we shall be doing that, but before we discover how to, we must be sure that we can do all the other operations properly.

Exercise 1 **a** Write the following in a shortened form:

i $(6 \sim 4)$
ii $(4 \sim 6)$
iii $(0 \sim 2)$
iv $(6 \sim 0)$

b Several of these integers are equal. Say which they are:

i $(6 \sim 3)$
ii $(3 \sim 6)$
iii $(6 \sim 0)$
iv $(3 \sim 0)$
v $(0 \sim 3)$
vi $(8 \sim 5)$

c Simplify, putting the answers into the alternative notation:

i $(6 \sim 4)+(6 \sim 3)$
ii $(4 \sim 6)+(0 \sim 2)$
iii $(3 \sim 6)+(6 \sim 0)$
iv $(8 \sim 5)+(5 \sim 8)$

Let us try subtraction by using the same rule as we have for addition. For instance, is it true that $(8 \sim 4)-(4 \sim 1) = (4 \sim 3) = (1 \sim 0)$? If we convert to the notation which we find simpler we get, $^{+}4-{}^{+}3 = {}^{+}1$ which is certainly true. Let us try $(6 \sim 4)-(4 \sim 1) = (2 \sim 3) = (0 \sim 1)$. In the other notation we get $^{+}2-{}^{+}3 = {}^{-}1$, which is also true. If this always works, we have found what we need.

Exercise 2 **a** Simplify each of the following by the method we have suggested and convert to the other notation to check the results.

i $(7 \sim 1)-(5 \sim 0)$
ii $(9 \sim 4)-(7 \sim 4)$
iii $(12 \sim 6)-(1 \sim 0)$
iv $(15 \sim 3)-(6 \sim 2)$
v $(14 \sim 0)-(8 \sim 0)$
vi $(15 \sim 12)-(8 \sim 3)$

Now check your work against the answers at the end of the section.

b Use the same methods with the following questions. Translate into the other notation and examine the results.

i $(7 \sim 2)-(2 \sim 1)$
ii $(9 \sim 1)-(4 \sim 2)$
iii $(17 \sim 12)-(3 \sim 7)$
iv $(6 \sim 5)-(5 \sim 4)$
v $(4 \sim 2)-(0 \sim 1)$
vi $(3 \sim 7)-(2 \sim 5)$
vii $(6 \sim 9)-(5 \sim 8)$
viii $(12 \sim 16)-(9 \sim 15)$
ix $(8 \sim 10)-(4 \sim 7)$
x $(7 \sim 7)-(0 \sim 6)$

Looking especially at the answer to number x in the last exercise, can you say what happens when you subtract a negative integer?

We can look at the question a different way. If you solve the equation $8-x=5$ you add x to each side giving $8=5+x$. Then, by subtracting 5 from each side, you write $8-5=x$. So $x=3$. Now try the same process with $5-x=8$.

$5-x=8 \Leftrightarrow 5=8+x \Leftrightarrow 5-8=x \Leftrightarrow x={}^{-}3$.

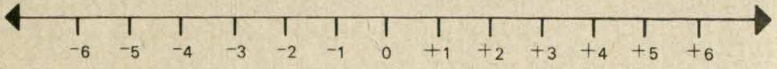
Now in $8-x=5$, you may replace the x by its solution set 3 and obtain the true statement $8-3=5$. Do the same with the other equation, $5-x=8$. The statement arrived at is $5-{}^{-}3=8$. If you read this as ${}^{+}5-{}^{-}3={}^{+}8$, does it agree with the results previously obtained? Is $(5\sim 0)-(0\sim 3)=(8\sim 0)$?

Exercise 3 **a** Compare ${}^{+}5-{}^{-}3={}^{+}8$ with $5+3=8$. What do you notice?

b Simplify the following:

i ${}^{+}6-{}^{-}3$
ii ${}^{+}7-{}^{-}4$
iii ${}^{+}2-{}^{-}5$
iv $0-{}^{-}6$
v ${}^{+}2-{}^{-}8$
vi ${}^{+}12-{}^{-}8$
vii ${}^{+}3-{}^{-}12$
viii ${}^{+}9-{}^{-}1$
ix ${}^{+}10-{}^{-}2$
x ${}^{+}10-{}^{-}20$

We can represent the negative and positive integers on the number line in this way:



Exercise 4 **a** What does the sign $>$ mean?

b On the number-line does $>$ mean 'is to the right of' or 'is to the left of'? (Look at the positive integers to be certain.)

c Say which of the following are true and which are false:

i ${}^{+}5>{}^{+}2$
ii ${}^{+}3>{}^{+}5$
iii ${}^{+}2>{}^{-}1$
iv ${}^{+}3>{}^{-}4$
v ${}^{-}1>{}^{-}3$
vi ${}^{-}6>{}^{-}4$
vii ${}^{-}4>{}^{-}6$
viii ${}^{-}3>{}^{-}1$
ix ${}^{-}6>{}^{-}2$
x ${}^{-}1>{}^{-}5$

d On the number line, to add is to move so many places to the right; e.g., $^{+}2+{}^{+}3$ means 'begin at $^{+}2$ and move three steps to the right', i.e., to $^{+}5$. Carry out this operation on the following:

i $^{+}1+{}^{+}4$
ii $^{-}2+{}^{+}5$
iii $^{-}3+{}^{+}5$
iv $^{-}3+{}^{+}2$
v $^{-}5+{}^{+}5$

e What do you notice about the result of $^{-}3+{}^{+}2$ and $^{+}2+{}^{-}3$? Is the same true of $^{-}6+{}^{+}4$ and $^{+}4+{}^{-}6$?

Exercise 5 **a** What does $<$ mean?

b On the number line, does $<$ mean 'is to the left of' or 'is to the right of'?

c Say which of these are true and which false:

i $^{+}2 < {}^{+}5$
ii $^{+}5 < {}^{+}4$
iii $^{-}1 < {}^{+}2$
iv $^{-}6 < {}^{-}3$
v $^{-}5 < {}^{-}1$
vi $^{-}5 < {}^{-}6$

d On the number line to subtract is to move so many places to the left. That is $^{+}7-{}^{+}4$ means 'start at seven and move four places to the left' which brings you to $^{+}3$. Try this operation on the following:

i $^{-}1-{}^{+}2$
ii $^{-}2-{}^{+}1$
iii $^{-}6-{}^{+}2$
iv $^{-}1-{}^{+}1$

e Would you get the same results from:

i $^{-}1-{}^{+}2$ and $^{-}2-{}^{+}1$
ii $^{-}6-{}^{+}2$ and $^{-}2-{}^{+}6$?

f Compare $0-{}^{+}1-{}^{+}2$ with $^{-}1-{}^{+}2$ and $^{-}2-{}^{+}1$. What do you notice?

Exercise 6 Simplify each of the following using either of your methods:

a $^{+}4-{}^{+}7$
b $^{+}6-{}^{+}12$
c $^{+}5-{}^{-}4$
d $^{+}12-{}^{-}6$
e $^{+}1-{}^{-}5$
f $^{-}2+{}^{+}4$
g $^{-}4+{}^{+}2$
h $^{+}2+{}^{-}4$
i $^{-}9+{}^{+}4$
j $^{-}4+{}^{-}6$
k $^{-}2+{}^{-}5$
l $^{-}1+{}^{-}9$
m $^{-}12+{}^{-}1$
n $^{-}10+{}^{-}10$

Now check that the other method also works. In terms of Exercises **4d** and **5d**, explain the meaning of the addition and subtraction of a negative integer.

Exercise 7 Find the solution sets of the following (in each case x is an integer). Note that the $^{+}$ sign has been omitted before positive integers.

a $\{x \mid x+12 = 3\}$
b $\{x \mid x-3 = 12\}$
c $\{x \mid x-12 = 3\}$
d $\{x \mid x-12 = {}^{-}3\}$
e $\{x \mid x+12 = {}^{-}3\}$
f $\{x \mid x-3 = {}^{-}12\}$

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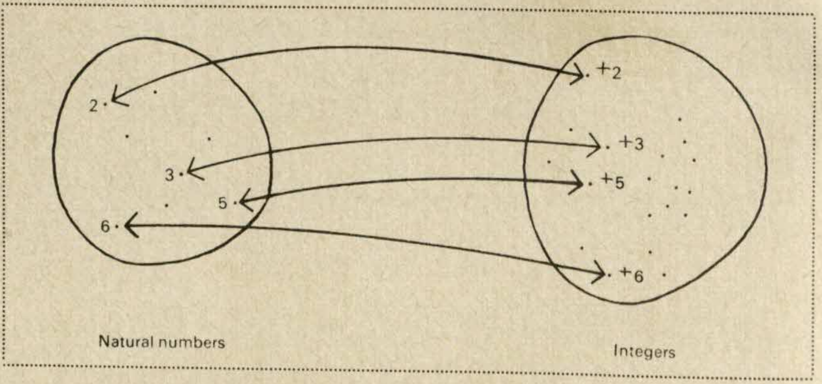
Can you now summarize the operations of addition and subtraction of integers? Adding on or subtracting a positive integer is like 'ordinary' addition and subtraction of natural numbers. Adding a negative integer has the same effect as subtracting a positive integer. Subtracting a negative integer has the same effect as adding a positive integer.
You will have noticed that there are two uses of the sign + and the sign − in this work. We have been writing them on different levels to distinguish them. What are the two uses? They are not always clearly distinguished; in fact it saves time often if we agree to leave out the $^+$ sign before positive integers. In the next exercise this will be done.

Exercise 8 Simplify the following:

a $7+-3$
b $7-3$
c $3+-7$
d $3-7$
e $7--3$
f $3--7$
g $7+3$
h $-3+-7$
i $-7+-3$
j $-7--3$
k $-7+3$
l $-3--7$
m $-3+7$

cation. We will now learn how to multiply both positive and negative integers.



The drawing shows a Venn diagram of the natural numbers and one of the integers. We have mapped some of the corresponding members of the two sets in one-to-one correspondence.
The images of 2 and 3 are $^+2$ and $^+3$. The product of 2 and 3 is 6 and the product of $^+2$ and $^+3$ is $^+6$. 6 and $^+6$ are also images of one another. There is still one important operation we have not studied: multipli- that the natural numbers are isomorphic to the positive integers.

Exercise 9 **a** Write down the image of the product in each of the following:
i 4×5 ii $^+6\times{}^+7$ iii 9×8 iv $^+4\times{}^+13$

b Write down the product of the images of the following:

i ${}^{+}6 \times {}^{+}7$ ii 5×4 iii ${}^{+}13 \times {}^{+}4$ iv 9×8

c Explain what you have just done in (a) and (b).

d What sort of integer is represented by $(2 \sim 0)$ and by $(0 \sim 2)$?

e Say what sorts of integers the following represent:

i $(3 \sim 6)$ ii $(4 \sim 3)$ iii $(7 \sim 9)$ iv $(9 \sim 7)$

f Express each of the integers in (e) in a simple form containing a 0.

g Write each of the integers of (e) in the simplest possible form.

h Which of the integers of (e) have corresponding natural numbers?

Now, we know that ${}^{+}2 \times {}^{+}3 = {}^{+}6$. We must find a way of working this out when the numbers are written $(2 \sim 0)$ and $(3 \sim 0)$ or $(4 \sim 2)$ and $(7 \sim 4)$.

Look at this diagram. It shows the way we can work out both of these. (Make sure our answers are correct.)

$$(2 \sim 0) \times (3 \sim 0) = (6+0 \sim 0+0) = (6 \sim 0)$$

$$(4 \sim 2) \times (7 \sim 4) = (28+8 \sim 14+16) = (36 \sim 30) = (6 \sim 0)$$

Now look at the following and see how they have been done:

$$\begin{aligned}
(2 \sim 1) \times (3 \sim 1) &= (6+1 \sim 3+2) &&= (7 \sim 5) = (2 \sim 0)\\
(4 \sim 2) \times (6 \sim 3) &= (24+6 \sim 12+12) &&= (30 \sim 24) = (6 \sim 0)\\
(2 \sim 0) \times (7 \sim 2) &= (14+0 \sim 0+4) &&= (14 \sim 4)\\
(13 \sim 7) \times (10 \sim 8) &= (130+56 \sim 70+104) &&= (186 \sim 174)\\
(1 \sim 2) \times (3 \sim 1) &= (3+2 \sim 6+1) &&= (5 \sim 7)
\end{aligned}$$

Exercise 10 Work out these as we did above and reduce them to the simpler form as we did with our first two above.

a $(2 \sim 4) \times (6 \sim 3)$

b $(2 \sim 0) \times (2 \sim 7)$

c $(1 \sim 2) \times (1 \sim 3)$

d $(4 \sim 2) \times (3 \sim 6)$

e $(2 \sim 4) \times (3 \sim 6)$

f $(7 \sim 10) \times (13 \sim 8)$

g $(7 \sim 13) \times (10 \sim 8)$

h $(7 \sim 13) \times (8 \sim 10)$

i $(5 \sim 6) \times (2 \sim 5)$

j $(5 \sim 5) \times (6 \sim 2)$

k $(1 \sim 1) \times (1 \sim 1)$

l $(4 \sim 1) \times (3 \sim 7)$

m $(0 \sim 4) \times (4 \sim 0)$

n $(0 \sim 4) \times (0 \sim 4)$

o $(0 \sim 4) \times (0 \sim 3)$

p $(0 \sim 4) \times (3 \sim 0)$

q $(4 \sim 0) \times (3 \sim 0)$

r $(4 \sim 0) \times (0 \sim 3)$

Exercise 11 Take each of the questions of the last Exercise and translate both questions and answers into the short form. (For example: rewrite $(2 \sim 4) \times (6 \sim 3)$ as ${}^{-}2 \times {}^{+}3 = \ldots.$)

Exercise 12 **a** Can you now find a rule for each of these?

positive integer × positive integer is a
positive integer × negative integer is a
negative integer × positive integer is a
negative integer × negative integer is a

b What do you think I might mean if I say: 'The product of like integers is a positive integer and the product of unlike integers is a negative integer'? Do you think this is a good way of putting it?

Exercise 13 Work out the following.

a ${}^{+}5 \times {}^{+}7$
b ${}^{-}6 \times {}^{+}9$
c ${}^{-}5 \times {}^{-}13$
d ${}^{+}17 \times {}^{+}9$
e ${}^{-}17 \times {}^{+}9$
f ${}^{-}17 \times {}^{-}9$
g ${}^{-}14 \times {}^{-}6$
h ${}^{-}12 \times {}^{+}14$
i ${}^{+}15 \times {}^{-}17$
j ${}^{+}45 \times {}^{+}8$
k ${}^{-}450 \times {}^{-}80$
l ${}^{-}250 \times {}^{+}45$

Exercise 14 Find solution sets for the following equations:

a $\{x \mid {}^{-}2 + x = {}^{+}6\}$
b $\{x \mid {}^{-}2 + x = {}^{-}6\}$
c $\{x \mid {}^{+}2 + x = {}^{-}6\}$
d $\{x \mid {}^{+}2 + x = {}^{+}6\}$
e $\{y \mid y + {}^{-}4 = {}^{+}12\}$
f $\{z \mid z + {}^{+}6 = {}^{+}18\}$
g $\{y \mid y + {}^{-}4 = {}^{-}6\}$
h $\{y \mid y - {}^{+}4 = {}^{-}6\}$
i $\{y \mid y + {}^{-}4 = {}^{+}6\}$
j $\{y \mid y - {}^{-}4 = {}^{-}6\}$
k $\{y \mid y - {}^{-}4 = {}^{+}6\}$
l $\{y \mid y - {}^{+}4 = {}^{-}6\}$

As we said before, it is often convenient to leave out + signs where they are not absolutely necessary. We often write the positive integers in the same way as the natural numbers. ${}^{+}7$ is then just 7. We also write $- {}^{+}7$ as -7 and since the value is the same $+ {}^{-}7$ is also written -7 sometimes.

Exercise 15 Write the following calculations in integers in a fuller notation and work them out.

a $15 - 20$
b $8 - 7$
c $24 - 57$
d $81 - 56$

Exercise 16 **a** Work out the following:

i $5 \times {}^{-}6$
ii ${}^{-}7 \times {}^{-}4$
iii ${}^{-}4 - {}^{-}4$
iv $6 - 3 \times {}^{-}4$
v $8 + 2 \times {}^{-}8$
vi ${}^{-}8 + 2 \times {}^{-}8$
vii ${}^{-}8 - 2 \times {}^{-}8$
viii ${}^{+}8 - 2 \times {}^{-}8$

b Find solution sets from the set of integers for the following:

i $\{x \mid x-4 = 3\}$
ii $\{x \mid x+4 = 3\}$
iii $\{x \mid 3x-4 = 11\}$
iv $\{x \mid 4x+6 = -2\}$
v $\{x \mid 8x-5 = -3\}$
vi $\{x \mid 4-2x = -2\}$
vii $\{x \mid 6+4x = 2\}$
viii $\left\{x \,\middle|\, \frac{4+x}{3} = 1\right\}$
ix $\left\{x \,\middle|\, \frac{2-x}{3} = 4\right\}$
x $\left\{x \,\middle|\, \frac{5-x}{2} = 1\right\}$
xi $\left\{x \,\middle|\, 5-\frac{x}{2} = 1\right\}$
xii $\left\{x \,\middle|\, 2-\frac{x}{3} = 4\right\}$
xiii $\left\{x \,\middle|\, 4+\frac{x}{3} = 1\right\}$
xiv $\left\{x \,\middle|\, \frac{2+3x}{7} = -1\right\}$
xv $\left\{x \,\middle|\, \frac{4-2x}{5} = 2\right\}$

c Find solution sets for the following, and draw graphs of them on a number line. In each, x is an integer.

i $\{x \mid -4 < x < 0\}$
ii $\{x \mid -4 < x \leqslant +4\}$
iii $\{x \mid -8 \leqslant x < -1\}$
iv $\{x \mid -7 \leqslant x \leqslant +2\}$
v $\{x \mid +12 > x > +7\}$
vi $\{x \mid -3 \leqslant x < +3\} \cap \{x \mid -5 < x < -1\}$
vii $\{x \mid +6 > x > -1\} \cup \{x \mid -2 < x < +3\}$
viii $\{x \mid -4 < x < -3\} \cup \{x \mid +4 < x < +5\}$

d Find the solution sets of the following:

i $\{x \mid {}^{-}2 \times x = {}^{+}6\}$
ii $\{x \mid {}^{-}2 \times x = {}^{-}6\}$
iii $\{x \mid {}^{+}2 \times x = {}^{-}6\}$
iv $\{x \mid {}^{\pm}2 \times x = {}^{+}6\}$
v $\{x \mid x \times {}^{-}4 = {}^{+}12\}$
vi $\{x \mid x \times {}^{+}6 = {}^{+}18\}$

Answers

Exercise 2a

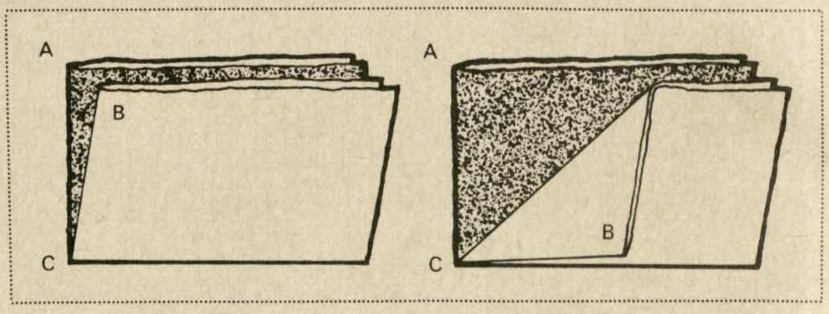
i ${}^{+}1$
ii ${}^{+}2$
iii ${}^{+}5$
iv ${}^{+}8$
v ${}^{+}6$
vi ${}^{-}2$

Section 13 **Experiments with paper and scissors**

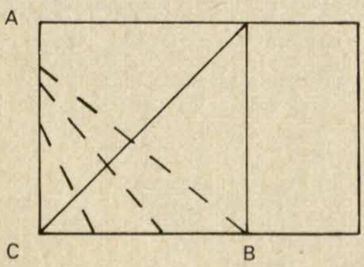
In this section you are going to carry out a number of experiments in paper- folding and cutting. You will need about thirty sheets of paper. Newspaper is excellent for the experimental work: a page of newspaper should be cut into four pieces; each of these will then be about the right size and thickness for the job.

You must not do this work in a haphazard way. You must learn to be systematic and to think. Try to predict the results before you get them, and to plan your discovery. You will then be able to answer the questions at the end of the section without difficulty.

Stage 1

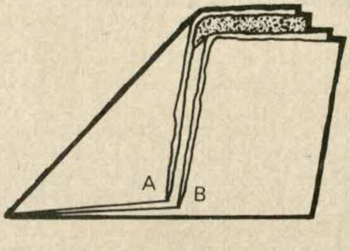


Fold four pieces of paper as is shown in the diagrams. (Fold once; fold again at right angles; fold one single fold down to the double fold.) The object is to cut off the corner where all the folds meet. The diagram will help you to see the different ways in which this may be done. Are

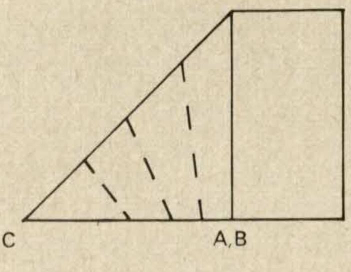


The dashed lines are suggested *straight* cuts there any others? Try as many as you can think of, but after each cut and before you open the paper out, try to decide what shape it will be. You will not always succeed in being right, but it is important to try. What is it that causes the results to be different?

Stage 2



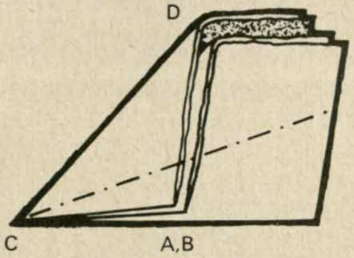
Fold at least five pieces of paper as in the diagram (as before, except that both corners are to be folded down this time. Again you will cut off the corner in as many different ways as you can, the object being to predict the results as far as possible. *Some* of the possibilities are suggested in the diagrams. Find the others. When do you get half the number of sides as thicknesses of paper? How can you get a completely regular figure? What properties of regular figures can you see from your cut-out? What are the differences between the regular figures and the star-shapes?



The dashed lines show three of the possible cuts

Stage 3

Fold once more along the chain line in the diagram. Cut off the corner again in as many ways as you can. There are at least five ways. Answer similar questions as in stage 2.



Bring D down to coincide with A and B
CD will then lie along CA and CB

Stage 4

Go on folding as before and observe the results until you find that the paper is too thick to cut any more. Try with each of these to cut a regular figure apart from any other variations you try. What do you notice about the figure you get from thirty-two thicknesses? (This is probably as far as you can get with scissors, but sixty-four thicknesses can be cut carefully with a fine saw; you can however manage as many as 128 thicknesses of fine tissue paper.) Do you think that if you could go on for ever folding and cutting, you would get nearer and nearer to another shape? Try cutting a thirty-two-fold piece of adhesive coloured paper into a regular figure. Then cut the figure into its triangular sectors. Can you stick them down in the shape of a rectangle? Could you find the area of the figure?

Stage 5

Cut out a circle of about three inches radius from adhesive coloured paper. Carefully mark it off into 10° sectors. How many will there be? Now cut out all these sectors. Do they look like triangles? Are they really triangles? Can you arrange them in the form of a rectangle? Is it exactly a rectangle? Can you find the approximate area of the rectangle and thus the circle?

Stage 6

Go back to your folding now and see if you can find a way to fold to get odd numbers of sides in your figures. Test your skill by folding to get seven sides or fifteen sides. Try also ten sides. It can be done, but it is difficult. If you can do it, you are very good!

Section 14 Numbers with direction: Positive and negative integers

Exercise 1 **a** Solve the equation: $13+x=25$.

b Solve the equation: $25+x=13$.

c The answer to (a) is a positive integer; what sort of number answers (b)?

d Work out: $-3+-2$ and $-2+-3$. Are the results the same?

e Find out if $(-4+-1)+-2=-4+(-1+-2)=(-4+-2)+-1$.

f Simplify the following:
i $-4--4$ ii $-5-+5$ iii $-7--9$

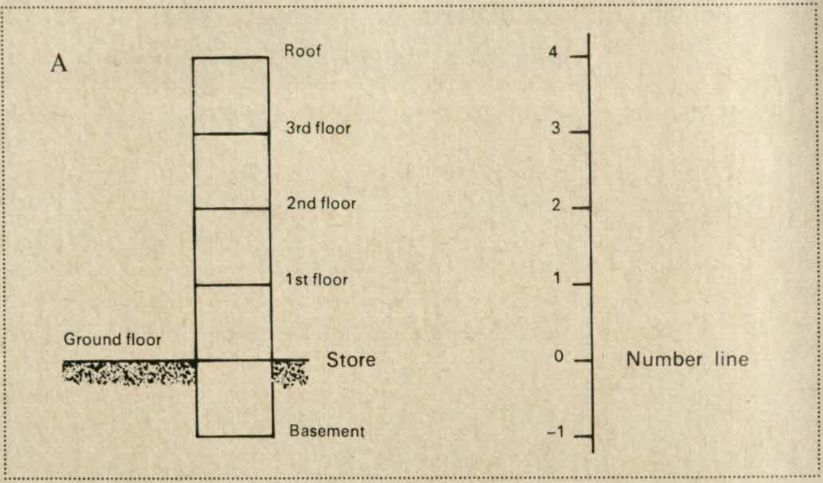
g Answer 'true' or 'false': A shepherd counted his flock. The total could have been
i any rational number
ii any natural number
iii any integer.

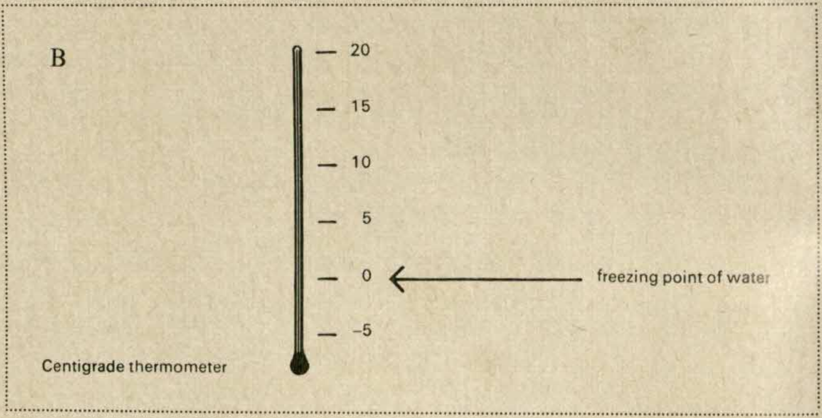
We use natural numbers (whole numbers) for counting things. If the things can be split up, like reams of paper, we may also need fractions (rational numbers) to count them. When we measure length and area we always need rational numbers available. As we have seen, positive integers behave exactly like natural numbers; this is why we often write them in precisely the same way. Negative integers cannot be used for counting most things: you cannot have -2 sheep or -5 reams of paper. You do sometimes meet with this sort of number, however, or at least with the idea behind negatives. Here are some examples.

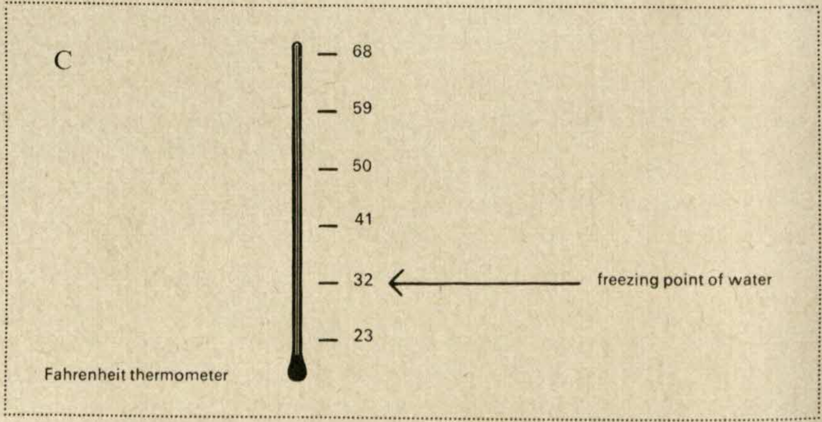
A If we number the levels of floors in a department store in accordance with a number line, we find the basement floor numbered -1.

B When we measure temperatures on the Centigrade scale, we are using a number line.

The thermometer illustrated reads up to 20°C.; the freezing point of water is 0°C. (This last statement is not necessarily exact, but this is how we usually work.) The temperatures which are sometimes given as so many 'degrees of frost' are also sometimes given as negative numbers. Five degrees of frost is thus -5°C.


A
Roof
3rd floor
2nd floor
1st floor
Ground floor
Store
Basement
4
3
2
1
0
–1
Number line


B
20
15
10
5
0
–5
freezing point of water
Centigrade thermometer


C
freezing point of water
Fahrenheit thermometer

C If the same thermometer had been marked in the Fahrenheit scale, it would be like the diagram. On this scale, 32° is the position allotted to the freezing point of water; 32 degrees of frost is then 0°F. Temperatures below this are given as negatives.

D If a man has a deposit of £100 in his bank and wishes to draw a cheque for £110, he can sometimes do this with the permission of the bank manager. If he does so, he really has a balance of £(–10) in the bank: he has to pay £10 in before he has nothing! It is the practice in banks to print these negative numbers in red on the statement of the account; the negative sign is then not needed. This, by the way, accounts for the saying about 'being in the red'.

Exercise 2 **a** If, in the store of example A, the boiler room is below the basement floor, what number would this correspond to on the vertical number line? How many floors up from the boiler room is the second floor?

b At Cambridge Bay in Victoria Island (in the Franklin District of Canada, north of the Arctic circle) the January temperature is about –30°C. In July, the temperature sometimes rises to 14°. By how many degrees has the temperature changed?

c A man has £142 in the bank. He draws cheques for £97 and £56. By how much is he in the red? The next day, he pays in a cheque for £12. What is his final balance?

Exercise 3 Say what sort of number you need (natural numbers, fractions, negative integers) for solving the following equations.

a $5x = 25$ **c** $7x+6 = 20$ **e** $3x+15 = 9$

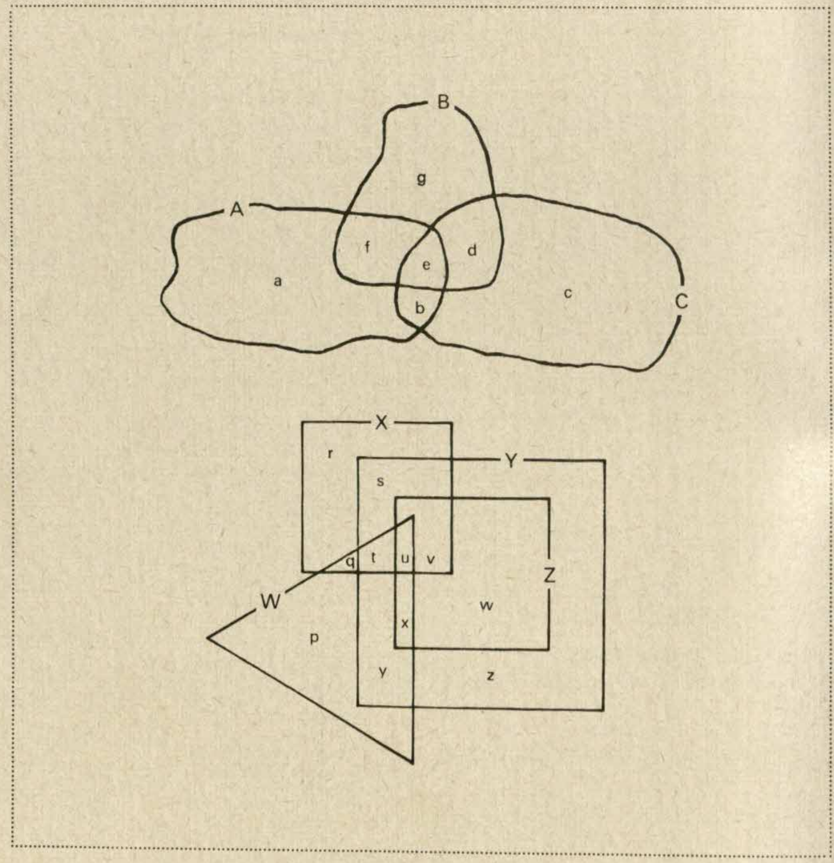
b $7x-6 = 35$ **d** $3x+2 = 15$ **f** $12-x = 17$

g Do we ever need negative fractions? Could they apply to any of the examples A, B, C or D above?

Section 15 **Some subsets of the plane**

Reminder
A. The plane is a special subset of space.
B. A line divides a plane into at least three subsets.

Exercise 1

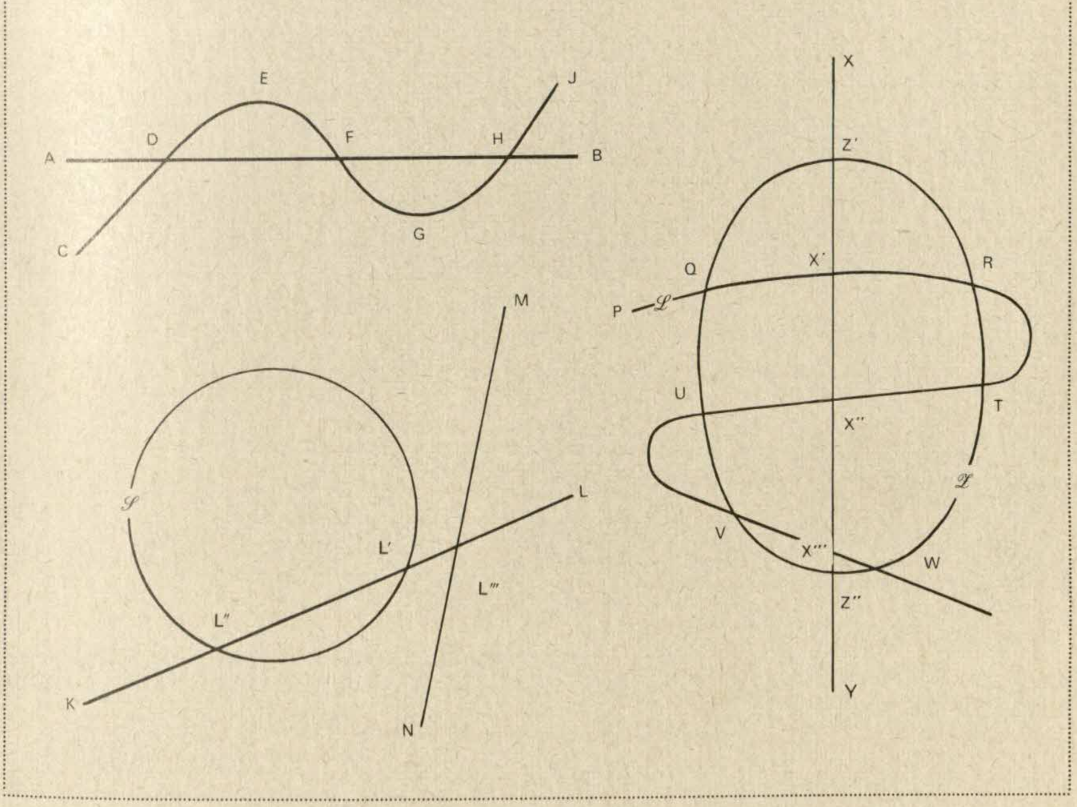


In the diagrams the letters A,B,C, W,X,Y, and Z are sets; the small letters represent regions. For each, answer by giving the small letters labelling the regions represented.

a i $A \cup B$
ii $A \cap C$
iii $(A \cap B) \cup C$
iv $(A \cup B) \cap C$
v $(A \cap C) \cap B$

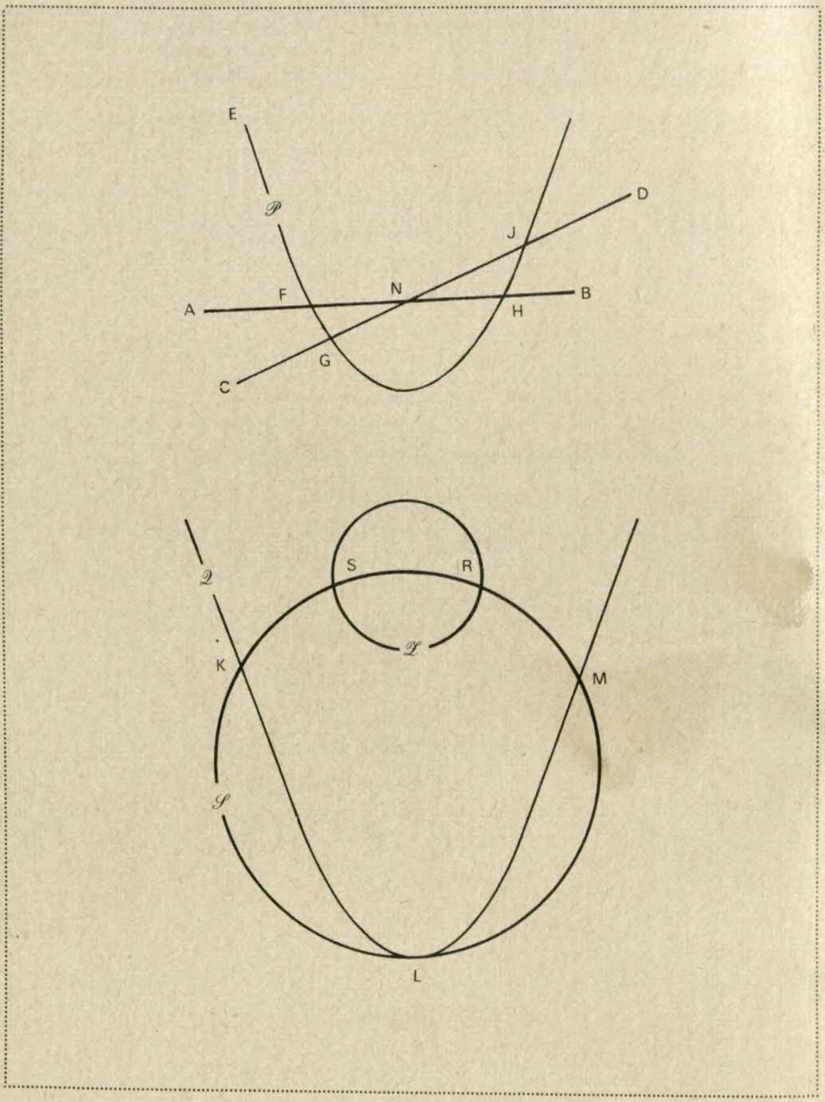
b i $X \cap Z$
ii $W \cup (Z \cup Y)$
iii $Z \cap Y$
iv $X \cap Y \cap Z$
v $W \cap Y$
vi $(W \cap X) \cup (Z \cap X)$
vii $(W \cup Z) \cap (W \cup X) \cap (X \cup Z) \cap (X \cup X)$

Exercise 2 In the following questions, lines are referred to by capital letters written in order. Closed curves are referred to by a single script letter only.



a Write down the intersection sets of the following:
i AB and CEGJ
ii MN and KL
iii $\mathscr{S}$ and LK
iv $\mathscr{S}$ and MN
v PRTUVW and $\mathscr{Z}$
vi XY and PRTUVW
vii XY and $\mathscr{Z}$

b



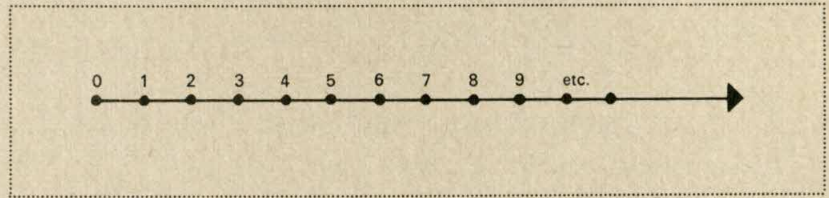
Write down the intersection sets of the following:

i AB and $\mathscr{P}$
ii CD and $\mathscr{P}$
iii AB and CD
iv $\mathscr{P}$ with the union of AB and CD
v $\mathscr{Q}$ and $\mathscr{S}$
vi $\mathscr{Q}$ and $\mathscr{T}$
vii $\mathscr{T}$ and $\mathscr{S}$

Section 16 **Numbers and number systems**

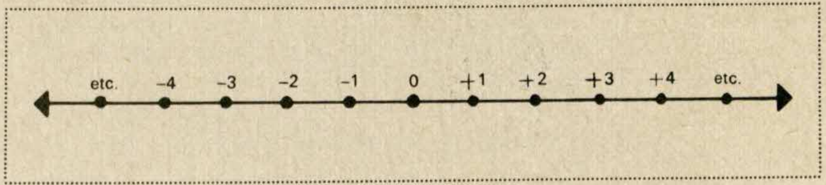
We have met several different sorts of numbers up to the present. Here is a summary of what we have discovered.

Numbers for counting
These are the natural numbers. We can occasionally count 0, so we must include zero in the natural numbers. If we leave out 0, the numbers are sometimes called 'whole numbers', but this is a term best avoided in future.



We can represent the natural numbers on the number line as separate points.

Numbers for counting rather special things
If we can have less than nothing, we need to be able to 'count' what we haven't got! This is where the integers come in useful. They can also be used to solve certain sorts of equations which cannot be solved in the natural number system. (e.g. $x+5 = 2$.) They are positive or negative, except for zero which is neither.



They are represented as separate points on the number line in this way.

Numbers for measuring
We need to be able to divide whole units of measurement into smaller pieces in order to improve the accuracy of measurement. The numbers we use to describe the measurements we make are rational numbers. They are formed by taking ratios of integers. Some ratios of integers turn out to be equal to other ratios so we have to be very careful about how we

construct such a number system ($\frac{2}{3} = \frac{4}{6}$). Rational numbers can also be positive or negative. These numbers crowd together on the number line so that it looks as if there is no space at all between them. In fact, there are still numbers on the number line which cannot be represented by the ratios of integers. The square root of 2 is an example but there are very many more such numbers.

Later on we shall learn more about these other numbers. If we take every point on the number line as representing a number, the set of numbers we get are called the REAL NUMBERS.

Exercise 1 Identify the sets of numbers needed to satisfy the following requirements.

a Counting a lot of separate objects.

b Measuring heights from the ground, when some of the measurements will need to go below ground level.

c $\{x \mid x-6 = 25\}$

d $\{(x,y) \mid x+y = 1000\}$ (the *whole* set)

e $\{x \mid x+25 = 6\}$

f $\{x \mid 3x = 25\}$

g $\{x \mid 3x+25 = 0\}$

While we were learning about these numbers, we have also been discovering some of the properties of numbers. There are very many properties which can be discovered, but some of these are much more important to us at the moment than are the others.

One of the things we discovered was that if you add two numbers or multiply two numbers, it does not matter which way round you do it. For instance, $4+5 = 5+4$. If you want to represent this in terms of *all* numbers we can 'generalize' by saying $a+b = b+a$. We should however say what kinds of things a and b can be. In fact a and b can be natural, integral, rational or real numbers and the statement is always true. In the language of sets, we can say that :

If $a \in \mathscr{E}$ and $b \in \mathscr{E}$, the $a+b = b+a$
when $\mathscr{E} = \{\text{natural numbers}\}$
or $\mathscr{E} = \{\text{rational numbers}\}$
or $\mathscr{E} = \{\text{integers}\}$
or $\mathscr{E} = \{\text{real numbers}\}$

Exercise 2 In all of the following, $\mathscr{E}$ will be understood as meaning the same set of possibilities as in the previous example.

a Would you agree that if $a \in \mathscr{E}$ and $b \in \mathscr{E}$, then $a \times b = b \times a$?

b If $a \in \mathscr{E}$, $b \in \mathscr{E}$ and $c \in \mathscr{E}$, is $(a+b)+c = a+(b+c)$?

c If $a \in \mathscr{E}$, $b \in \mathscr{E}$ and $c \in \mathscr{E}$, is $(a \times b) \times c = a \times (b \times c)$?

d If $x \in \mathscr{E}$, $y \in \mathscr{E}$ and $z \in \mathscr{E}$, is it true that always $x \times (y+z) = x \times y + x \times z$?

If you are in doubt about any of these you may try numbers in the expressions. But remember that you have not *proved* anything about the numbers you have not tried.

The laws you have been looking at here are the fundamental ones of arithmetic. Because they are so important they have special names:

the COMMUTATIVE LAWS: for addition: $a+b = b+a$
for multiplication: $a \times b = b \times a$
the ASSOCIATIVE LAWS: for addition: $a+(b+c) = (a+b)+c$
for multiplication: $a \times (b \times c) = (a \times b) \times c$
the DISTRIBUTIVE LAW: for multiplication over addition: $a \times (b+c) = a \times b + a \times c$.

For practical purposes, the associative laws mean we can add (or subtract) and multiply (or divide) in any order we please, provided we do not mix up addition and multiplication: this is where the distributive law applies.

Exercise 3 **a** Simplify by the simplest method; explain at each stage which laws you are using:

i $104+27+16+13$
ii 3s. 8d. + 9s. 2d. + 1s. 4d. + 2s. 10d.
iii $2 \times 17 \times 5 \times 3$
iv $4 \times 68{\cdot}69 \times 5^2$
v $31 \times 27{\cdot}32 + 22{\cdot}68 \times 31$

b Use the distributive law to work out:

i 67×40 ii 94×37

c Explain how the distributive law is used in the normal layout for multiplication of numbers of more than one digit.

Now a word about the numbers systems we have been discussing. Some time ago we agreed to write the rational numbers with 1 on the bottom (denominator 1) as the corresponding natural numbers. For instance, we said we would write $\frac{5}{1}$ as 5. After all, it behaves exactly the same way. Later, we said that the positive integers would be written without the + sign; $^{+}5$ would be written as 5, because it has the same kind of properties. We are even prepared to change certain numbers: if we find in a calculation the instruction 'subtract minus three', written '$-{}^{-}3$', we are happy to change it to something which has the same effect: $+{}^{+}3$ or, more simply, $+3$.

It is not true, however, that $\frac{5}{1}$ is the same number as 5 or ${}^{+}5$. They belong to three different number systems. The 'biggest' of the number systems, the real numbers, does contain elements which can do all the work of all the numbers in the other systems. These elements are said to be ISOMORPHIC with the elements of the other systems.

If this confuses you, perhaps an analogy will help you to grasp this difficult idea. Imagine a man who owns two cars: both the same model and the same colour—they differ in appearance only in the registration number. When he travels to work he uses one car or the other, and whichever he uses, the effect is exactly the same and even costs the same. This does not however alter the fact that his vehicles are not the *same*. The cars might almost be said to be isomorphic! In the same way $\frac{9}{1}$, ${}^{+}9$, and 9 have the same effect but are really different.

Section 17 **Transformations**

In this section we shall read '→' as 'changes to' or 'becomes' so that '$5 \rightarrow 8$' means '5 changes to (or becomes) 8' and '$x \rightarrow x+3$' means 'x becomes $x+3$'.
On the number line this transformation means that every point is moved along three steps to the right. This kind of transformation is called TRANSLATION: every point is moved the same amount in the same direction.

Exercise 1 **a** Using the translation $x \rightarrow x+3$, complete the following statements:

i $15 \rightarrow$
ii $\quad \rightarrow 21$
iii $2\frac{1}{2} \rightarrow$
iv $\quad \rightarrow 3+4\frac{1}{2}$
v $2\frac{1}{4}+3\frac{5}{8} \rightarrow$
vi $\quad \rightarrow 3{\cdot}45$
vii $\quad \rightarrow 12{\cdot}41-2{\cdot}39$
viii $3\frac{1}{5}\times 3\frac{3}{4} \rightarrow$
ix $\quad \rightarrow 2{\cdot}25\times 1{\cdot}5$

b Using the translation $y \rightarrow y-4$ complete the following:

i $4 \rightarrow$
ii $12{\cdot}25 \rightarrow$
iii $\quad \rightarrow 3$
iv $3 \rightarrow$
v $-2 \rightarrow$
vi $-2{\cdot}25 \rightarrow$
vii $\quad \rightarrow -2{\cdot}25$
viii $\quad \rightarrow -5\frac{1}{8}$

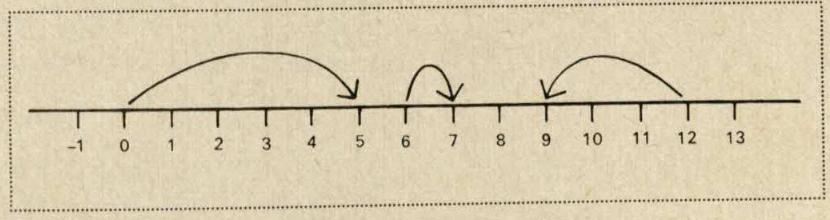
c Using the translation $z \rightarrow 3z$, complete the following:

i $2\frac{1}{3} \rightarrow$
ii $\quad \rightarrow \frac{5}{8}$
iii $\quad \rightarrow 2\frac{3}{7}$
iv $4\frac{1}{3} \rightarrow$
v $0 \rightarrow$

Exercise 2 **a** Using another type of transformation $p \rightarrow \dfrac{p}{3}+5$ complete the following:

i $9 \rightarrow$
ii $\quad \rightarrow 7\frac{2}{3}$
iii $7 \rightarrow$
iv $\quad \rightarrow 7$
v $0 \rightarrow$
vi $\quad \rightarrow 6$
vii $12 \rightarrow$

b



Illustrate all the results of (a) on a number line, in the way we have begun above. Copy the diagram and complete it.

i Are all the jumps of equal size?

ii Do they form a pattern?

iii Do you think there is a number which would be transformed into itself? Can you find it? Can you also find it by solving an equation?

c Work out a number of transformations using $x \rightarrow \frac{x}{4} + 3$ by completing the table.

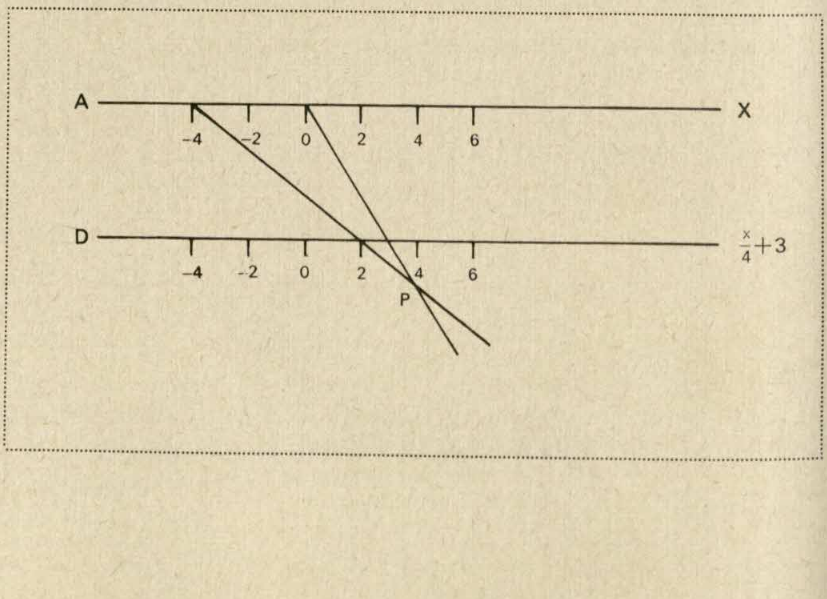
x	-4	-2	0	2	4	6	8	10
$\frac{x}{4} + 3$	2		3					

Next, draw a large diagram of two parallel lines 3 cm. apart and 14 cm. long. Mark them as number lines from about -4 to 10. Carefully draw in rays to represent the transformations in your table. Two have been done for you in the sketch. What sort of line shows the 'invariant' (unchanging) number?

Now draw in two more parallels at 1 in. distances between the first two lines and mark them off as number lines too. Call the four lines A, B, C, and D from the top downwards. The transformation represented by A → D is $x \rightarrow \frac{x}{4} + 3$. Can you find the transformations represented by B → D and C → D?

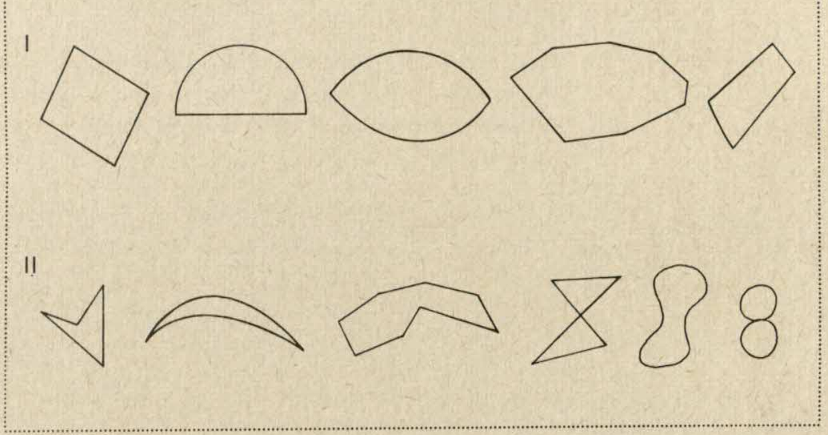
You will find it easiest to look at what happens to 0 in the transformation and then to consider, say, 4.

If you were to draw another line E through P parallel to D, what would be the transformations A → E, B → E etc.?



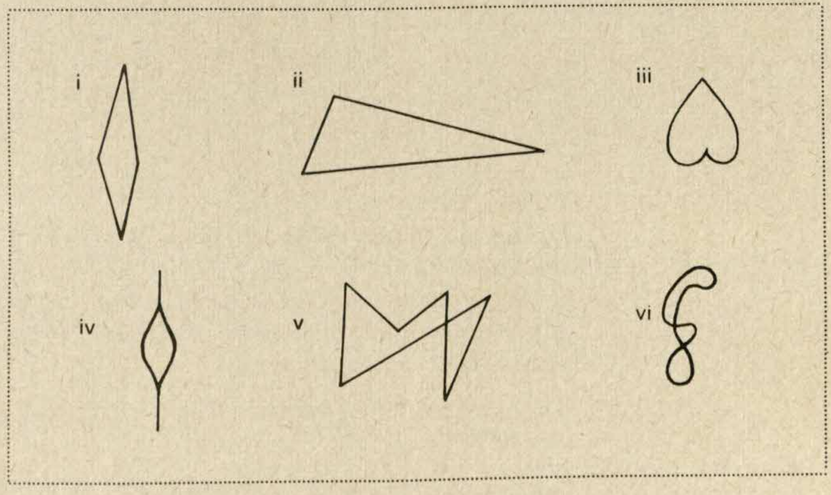
Section 18 **Convexity**

Exercise 1 **a**



Compare the preceding two sets of figures and see if you can discover why they form sets with a common property.

b



c Look at these figures and decide into which set they should be placed.

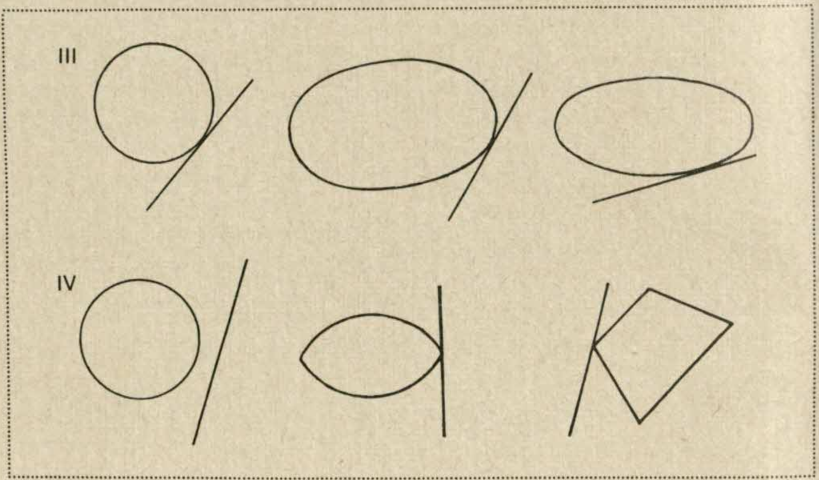
The set I in (a) is a set of CONVEX figures. Draw a few convex figures and a few non-convex ones too. Shade the interior of each convex figure.

d By looking at any of the convex figures you have before you, see if you can find two points P and Q in any of them such that the straight segment PQ lies at least partly outside the figure. (Regard the figure as the *closed* set of the interior, that is, as including the boundary.)

e Choose a point inside a convex figure and consider what will happen to *any* straight line drawn through that point. How many times will it cut the boundary of the figure?

f Draw a convex figure and a straight line so that the whole of the figure lies on one side of the line. Such a line is called a 'support line'. Can you now construct a support line so that at least one point of the curve lies in the line set itself?

Exercise 2



Examine the two sets of figures:

The first set consists of convex figures with support lines of a kind which are called TANGENTS; the support line in each example cuts the convex figure in one point only. The second set is a set of convex figures with support lines which are not tangents; some of these also cut the convex figure in one point only.

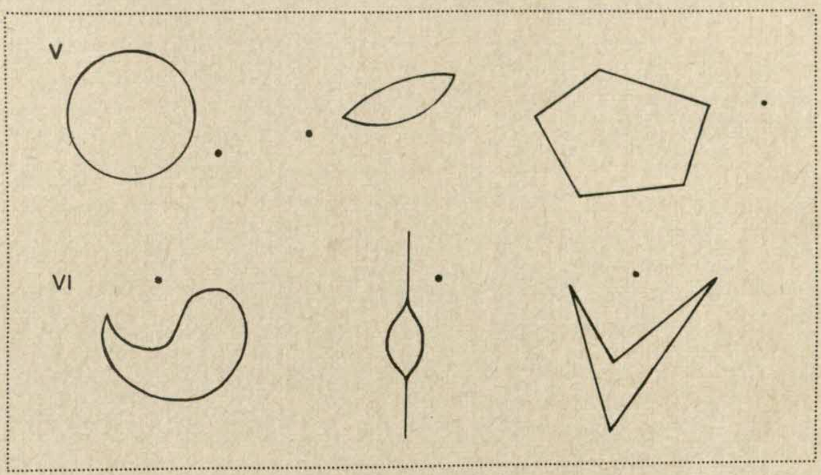
a Draw three convex figures and support lines to continue each of the sets III and IV.

b Can you find any point on the boundary of a convex figure which you have drawn or which is drawn in the book such that you cannot draw a support line through it?

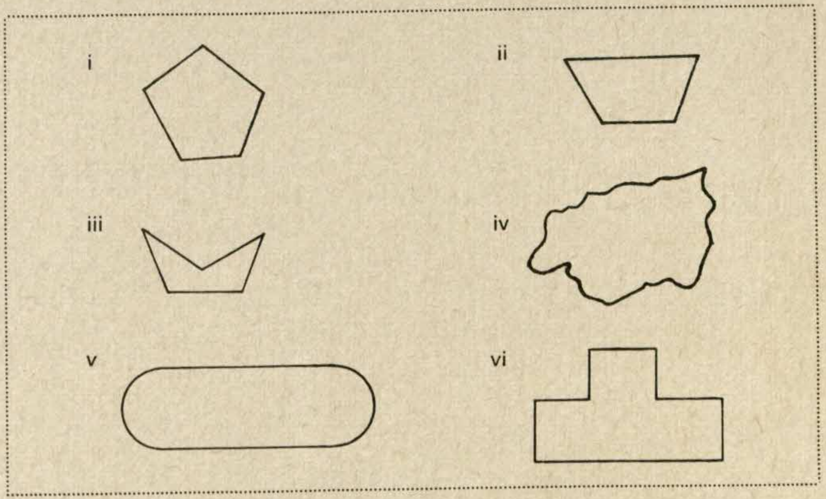
c Draw a non-convex figure. Can you find at least one point on the boundary where you cannot draw a support line?

d Is there only *one* support line through all the points on the boundary of a convex figure? See if you can find an example with more than one at some point.

Exercise 3 **a**



The two sets V and VI show respectively convex and non-convex figures together with an exterior point. Examine each of the diagrams and think about the shortest distances from the exterior points to the figures. Would you be able to use this idea to distinguish convex figures from concave?



Examine the preceding figures and decide whether they are convex or not.

Exercise 4

a Draw several illustrations of line sets which divide a plane into three subsets, (two half-planes and the line sets themselves).

b Can you use the idea of Exercise 3 to say which regions are convex and which concave?

c Is an infinite straight line drawn in an infinite plane concave or convex?

Reinforcement Exercise 3

Solve the following equations:

a
i $x+13 = 21$
ii $x-13 = 21$
iii $3x = 57$
iv $3x+4 = 73$
v $4x-5 = 63$
vi $\frac{x}{4} = 14$
vii $\frac{3x}{5} = 27$
viii $\frac{4x+5}{11} = 3$
ix $\frac{5x-6}{13} = 3$
x $\frac{7x-2}{11}+3 = 6$
xi $\frac{3x+4}{2}-5 = 18$
xii $3-\frac{3x+4}{5} = 1$

b
i $2x+3x = 4$
ii $2x+4 = 3x$
iii $3(4x-2) = 18$
iv $3(2x+4) = 5$
v $3(2x+4) = 4x$
vi $3(2x-5) = 5x$
vii $3(4-3x) = 7x$
viii $3(4-3x) = 7$
ix $\frac{2(3x-4)}{8} = 2$
x $\frac{3(4-2x)}{4} = 6$
xi $\frac{3(4-2x)}{7} = 4x$
xii $\frac{3(4-2x)}{7}+1 = -\frac{5}{2}$

c
i $\frac{x}{3}+\frac{x}{2} = 5$
ii $\frac{2x}{3}+\frac{3x}{2} = 13$
iii $\frac{1}{x}+2 = 7$
iv $2-\frac{1}{x} = 4$
v $\frac{1}{x}-\frac{2}{x} = 7$
vi $4(2x-4) = -x$
vii $4x-5 = 1$
viii $5x = 17$
ix $\frac{6x+21}{3}-5 = -4$
x $4(3x-5) = 12$
xi $x-17 = 34$
xii $\frac{3x}{7} = 2$

d
i $3x-5x = 2$
ii $4(3x+5) = 5$
iii $x+17 = 5$
iv $3x+5 = -1$
v $\frac{4x+9}{3} = -1$
vi $\frac{5x-4}{7}+2 = 0$
vii $5(3x-2) = 4x$

Answers on page 233.

Section 19 **Factors**

Exercise 1 For this exercise and the next we are using only positive integers.

a Write out the set $A = \{\text{divisors of } 12\}$.

b Examine the following subsets of A:
$B = \{12, 1\}$; $C = \{6, 2\}$; $D = \{4, 3\}$.
What do you notice about the elements or members of these sets.

Each of B, C, and D are sets of pairs of FACTORS of 12.

c Write down all the sets of pairs of factors of 20.

d Write down the prime factors of the following numbers (numbers may be repeated):

i 16 ii 9 iii 11 iv 21

e What do you know of the numbers 16 and 9? How can you discover this from your sets of factors.

f What sort of number is 11? How can you see this from your factors?

Exercise 2 **a** Using your set D from Exercise 1, we can write the following product $2 \times 2 \times 3$. What is this product?

b Are these numbers prime or not?

c Write down the set of even primes.

d Write down the prime factors of i 18, ii 20, iii 56.

Exercise 3 $F = \{9, 1\frac{1}{3}\}$ $G = \{18, \frac{2}{3}\}$ $H = \{20, \frac{3}{5}\}$

Write down a description for sets B, C, D, F, G, H as well as many other sets. How many sets will your description describe altogether?

Exercise 4 **a** What are the two factors of

i 5×17 ii 51 iii $2a$ iv ab?

b Write down three sets of two factors for each of the following expressions:

i $2ab$ ii $6b^2$ iii $12a$

c Write down two factors of $(a+b)\times(c+d)$.

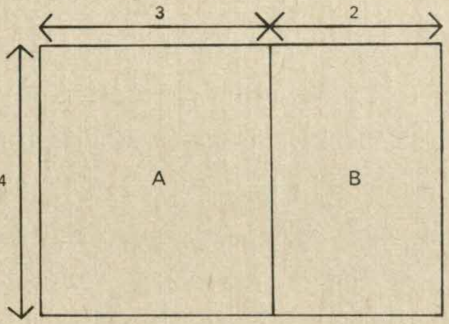
d Write down two factors of $7\times(e+f)$.

e Write down two factors of $a\times(g+h)$.

f Write down three factors of $2a\times(j+k)$.

g Write down four factors of $ab\times(c+d)\times(e+f)$.

Exercise 5



Write down the areas of rectangle A and rectangle B in square units.

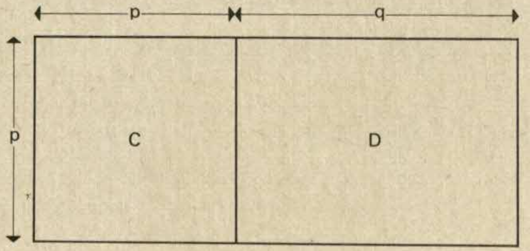
a How long is the whole rectangle? How wide is it?

b What is the area of the whole rectangle?

c What is area A plus area B?

d Why are the results of (b) and (c) related?

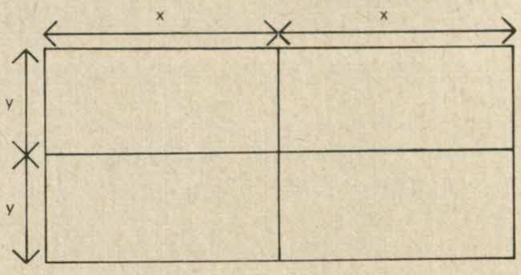
Exercise 6 **a**



Write down the areas of C, D and the whole rectangle.

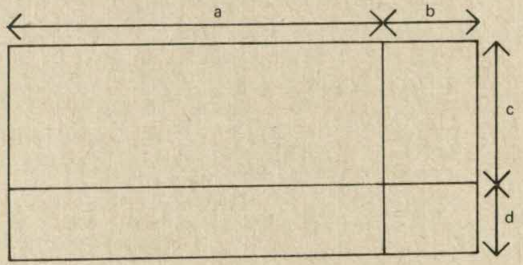
b What is the relationship between these three quantities? Write it down in mathematical symbols.
Check what you have written now before going on.

c



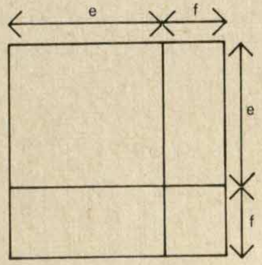
Work out a similar relationship with the diagram 6(c)

d

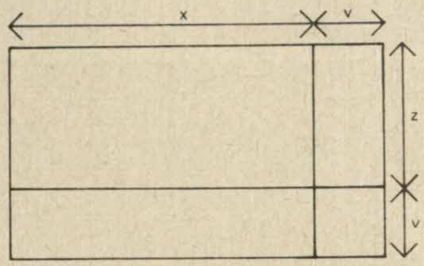


Can you find an expression equal to $ac+bc+ad+db$?

e Can you use this diagram to find an expression equal to $(e+f)\times(e+f)$?



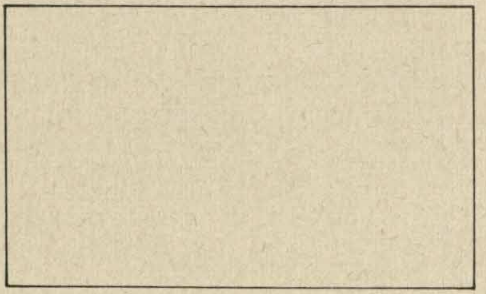
f



Can you find an expression equivalent to $(x+y) \times (z+y)$?

g You have been writing down statements of equivalence in parts (b), (c), (d), (e), (f). Are the statements always true whatever the values of p, q, r, x, y, a, b, c, d, e, and f? How do you know?

Exercise 7



A rectangle like this one cut out of card was given to each member of a class.

a Some members of the class measured their cards very carefully and obtained a very accurate result. They then calculated the area. What result did they get?

b One boy measured the length correctly but read the width as $1\frac{9}{16}$ in. How much too big was the area he calculated?

c Two girls measured the length as $2\frac{9}{16}$ in. and the width correctly. How much too big were their calculated areas?

d One boy measured both the length and the width $\frac{1}{16}$ in. too big. When he calculated the area of his rectangle he arrived at an answer which was too big by one of the following: which?

i $(\frac{1}{16}\times\frac{1}{16})$ sq. in.
ii $(\frac{1}{16}\times 1\frac{1}{2}+\frac{1}{16}\times\frac{1}{16})$ sq. in.
iii $(\frac{1}{16}\times 2\frac{1}{2}+\frac{1}{16}\times 1\frac{1}{2}+(\frac{1}{16})^2)$ sq. in.

e Explain how the last result is connected with Exercise **6f.**

f The class now copied the card cut out themselves by drawing it in their books. The smallest measurements which any member of the class made were $\frac{1}{8}$ in. too short for the length and the width; the largest were $\frac{1}{16}$ in. too long. Show how you can use the result of 6(f) to find the difference between the areas of the smallest possible drawing and the largest.

Exercise 8 **a** Sketch diagrams which show equivalent expressions (*expansions*) of each of the following:

i $(a+b)^2$
ii $(a+2b)^2$
iii $(2a+b)^2$
iv $(2a+2b)^2$
v $2(a+b)^2$
vi $4(a+b)^2$
vii $2z(x+y)$
viii $(x+y+z)(x+y)$
ix $(x+2y+z)(2x+y)$
x $(x+y+z)^2$

b Some people have a rule for finding the square of an expression. It is:

square and square and square . . .
and twice the product of every pair.

What do you think this means? (Look at **a**[i], [ii], [iii], [iv], [x].)

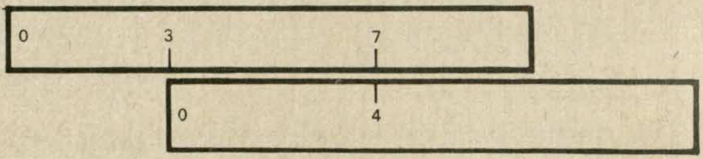
c Rewrite the following expressions in an equivalent form without using diagrams. You have seen expressions like these before.

i $(x+y)^2$
ii $x^2+4xy+4y^2$
iii $2x^2+4xy+2y^2$
iv $2a(b+c)$
v $3ab+3ac$
vi $5xy+5xz$

Section 20 **A tool for calculating**

For the first two exercises you will need four strips of thin card 1 in. wide and about 10 in. long, and two about 20 in. long.

Exercise 1 Write a number 0 at the left hand edge of each of two cards. Three inches from the 0 on one of them write 3 and mark the exact position on the lower edge. On the upper edge of the second card mark and number the 4 in. position. Place the two cards together as in the diagram.



Opposite the 4, mark on the upper card the value 7. Measure the distance of the 7 from the left hand edge. What did you find?

Now on each card mark all the 1 in. positions and number them. Place the cards together again as before. Can you now read off the values of $3+4$, $3+5$, $3+2$, $3+7$ etc.?

Move the lower card so that the 0 corresponds with 4 on the upper. Can you now read off the values of $4+1$, $4+3$, $4+6$, etc.?

You may like to think of what you have just made as a transformations machine. It maps the numbers on the lower card into the same number plus something, the result of the transformation appearing on the upper card. If the lower one is set against 5 on the upper, the transformation is $x \rightarrow x+5$ where x is any value on the lower card. (What could you do about $x \rightarrow x+5$ where $x \geqslant 6$?) Write down the transformation for the position when the 0 of the lower card is against a value y on the upper.

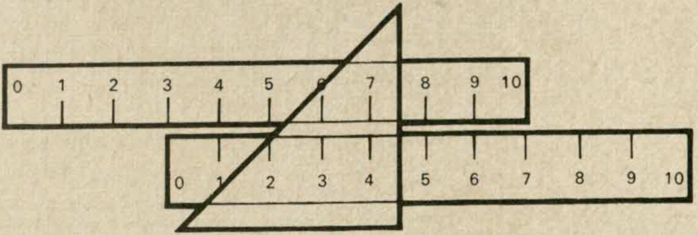
You have just made a very simple tool for adding numbers. What would you have to do to include (i) fractions and (ii) negative integers?

Make new strips to do both of these jobs and check that they are working properly. (You will probably find it an advantage to use much longer strips for the negative case.)

Exercise 2 Set up your strips again for addition of numbers to 3. Can you use this set up for establishing the answers to the following?

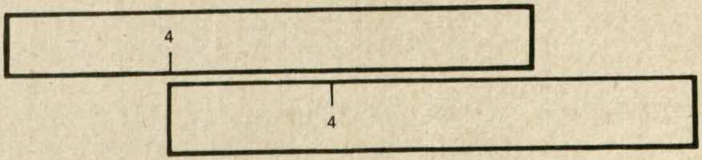
i $8-5$
ii $9-6$
iii $4\frac{5}{8}-1\frac{5}{8}$
iv $8-3$
v $7-4$
vi $9-3$
vii $6{\cdot}5-3{\cdot}5$

Is there any difference between the operation of the tool for addition and subtraction? How many additions and subtractions does the set-up give in each position? How many positions are there? What you have been making is a slide-rule for addition and subtraction. This is not the usual way of using a slide rule for the processes are easy enough to do in simple cases like this. The principle is very important however. There is one addition you could make to your 'slide rule' and that is a line selector to help you find the corresponding position on the two strips. The illustration shows how you might use a set square to do this. There is something wrong with the drawing. Can you see what it is?

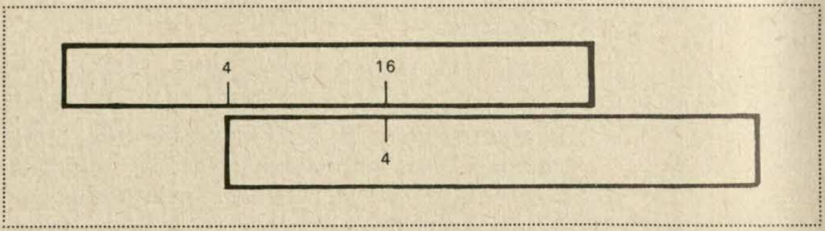


We have just made a tool for carrying out addition. We are going to carry on now to see if we can do the same sort of thing for multiplication. You will need two strips of plain card and two sheets of strong graph paper. Cut the plain strips $\frac{1}{2}$ in. wide and at least 5 in. long.

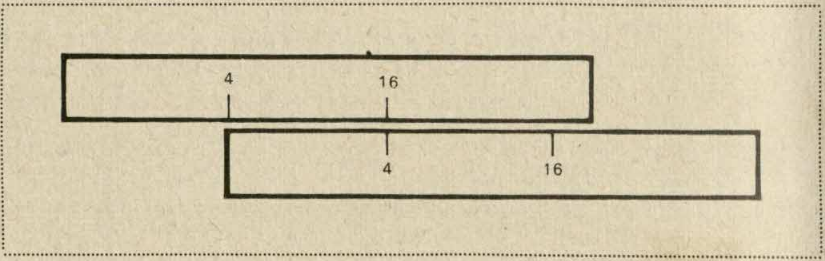
Exercise 3 a Select a small number between 3 and 8 and mark each strip as we have shown.
We chose 4 as the number but you may use another.



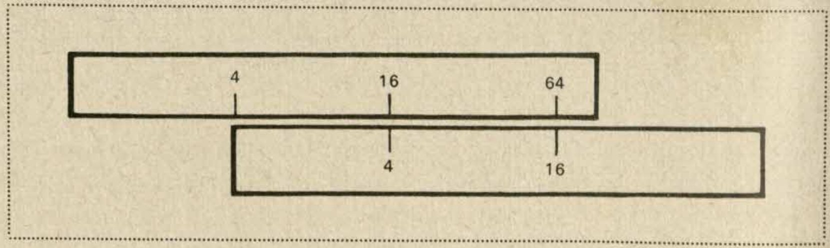
Now place the strips as we have done. We shall use the strips for multiplying: 4 × 4 = 16. So we shall mark the position of 16 as shown.



We can now put in 16 on the upper strip



and read off 4 × 16 = 64, marking the 64 on the upper strip.



Do exactly the same with your strips but see that the numbers you put in are the right ones for your starting point. Carry on now marking numbers to the right until you cannot get any more in.

b If we try to put intermediate numbers what shall we find as we look along the line to the right?

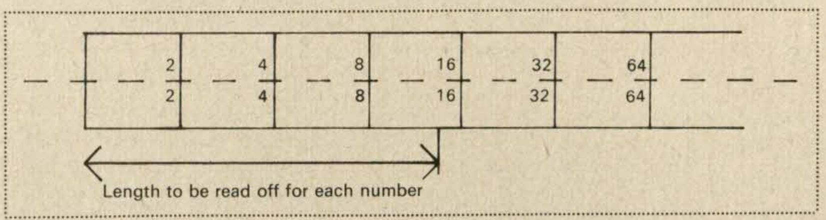
c What number should go at the beginning of the strip, 0 or 1? Decide which by trying out multiplication by 0 and 1.

Exercise 4 **a** So that you can get in intermediate numbers more easily, make a new slide rule from graph paper. Cut a strip 7 in. long and 1 in. wide. See that you cut exactly along the heavy lines on the paper.
Now mark the paper strip as we have done in the diagram.

Next, cut along the dotted line carefully.
Does the slide rule you have made work for multiplication?

b Now by experiment, trial, and error, see if you can put in some of the intermediate numbers correctly. You could try to put in 3 first and then 5, 6, 7, 9, etc.

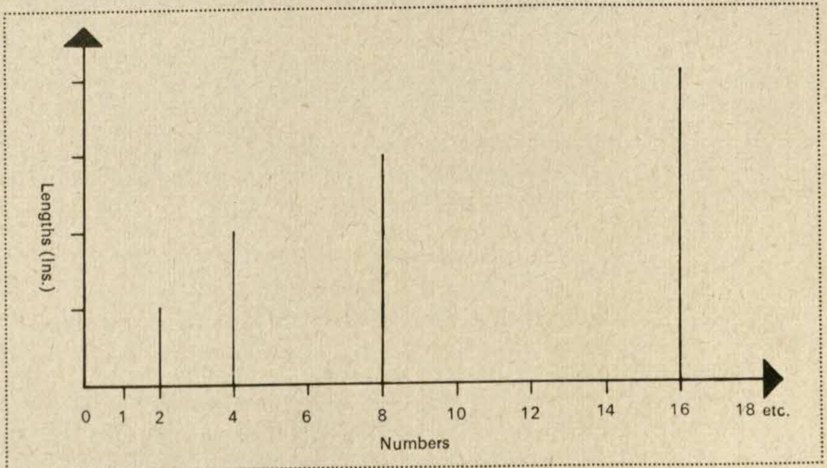
c Now read off the lengths used for these numbers. Read from the left hand edge of the strip as we have shown:



Make a table of your lengths:

number	length in inches (from LHS)
1	
2	
3	
4	
5	
etc.	

d



That last step was not easy to get done accurately, was it? Here is an easier way of finding the correct lengths, but you still have to be very careful in doing it.

On your second sheet of graph paper mark a horizontal axis and a vertical axis as we have done in the diagram.

The horizontal axis is for the numbers which you are intending to write on your slide rule; the vertical axis is for the lengths which stand for these numbers. Put in those which you already know, as in the diagram.

Now draw a smooth curve through the ends of the vertical lines. You can now read off the lengths for other values.

e Check the table you made in (c) above; you can compare the values in it with the values from your graph. If your slide rule was not quite right, you can now make a much better one which works accurately. Will it be absolutely accurate? For instance will it help you to work out $6{\cdot}45 \times 7{\cdot}32$ exactly? How accurate could you expect it to be?

f Use your slide rule for the following calculations to see if it is correct now.

i $2 \times 8 = 16$
ii $3 \times 3 = 9$
iii $2 \times 6 = 12$
iv $12 \div 4 = 3$
v $12 \div 3 = 4$
vi $10 \div 2 = 5$
vii $10 \div 5 = 2$
viii $15 \div 3 = 5$
ix $15 \div 5 = 3$

Look at your graph. The length of the vertical line corresponding to each number is called the LOGARITHM of the number. For shortness we often write 'log' instead of 'logarithm'.

Exercise 5 **a** Check from your slide rule that:

i $\log 2 + \log 8 = \log 16$
ii $\log 10 - \log 5 = \log 2$

b Complete the following:

i $\log 3 + \log 4 = \log$
ii $\log 2 + \log 5 = \log$
iii $\log 5 + \log 3 = \log$
iv $\log 4 + \log \quad = \log 12$
v $\log 12 - \log 3 = \log$
vi $\log 10 - \log 2 = \log$
vii $\log 15 - \log 3 = \log$
viii $\log 9 - \log 3 = \log$

c Fill in the missing words in this statement: 'The multiplication of numbers becomes the — of their logarithms; the division of numbers becomes the — of their logarithms.'

On our slide rule we used 1 unit of length for the number 2 and the numbers at equal distances were the powers of 2. Our logarithms thus depended on 2; they are said to be 'to the base 2'. Slide rules which we buy are made to the base 10, but the idea is the same.

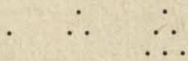
Exercise 6 Here is a table for you to complete. You do not need your slide rule to do it. It is a 'translation' from calculations with numbers to calculations with logarithms.

	numbers	logarithms
a	4 7 =	log 4 + log 7 = log
b	5 8 =	log + log = log
c	3 × 14 =	log log = log
d	7 =	log log = log 21
e	2·5 3·5 =	log log = log
f	24 ÷ =	log − log 3 = log
g	=	log 15 − log 5 = log
h	18·75 ÷ =	log log = log 7·5

The idea of logarithms was discovered by a Scotsman, John Napier. He wrote his book *Mirifici logarithmorum canonis descriptio* and published it together with the first set of tables in 1614. He had been working for many years on the subject and had written another book on logarithms, but this was not published until after his death. He died in 1617 at the age of 67. He was a contemporary of Shakespeare.

Section 21 **Pythagoras' theorem**

In Book 1 we saw that dots representing certain numbers could be arranged in triangles. The sequence of numbers with this property is:



1 3 6 etc. These numbers are called 'triangular numbers'.

Exercise 1 **a** Carry on the series of triangular numbers to at least ten terms.

b Examine the following table:

$$\begin{array}{l}
0\times6+1+\ 0=1^3=\ 1\\
1\times6+1+\ 1=2^3=\ 8\\
3\times6+1+\ 8=3^3=27\\
6\times6+1+27=\dots\dots\dots\\
10\times6+1+\dots\dots\dots\dots\dots\\
15\times6+1+\dots\dots\dots\dots\dots\\
21\times6\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

Where are the triangular numbers? How is the column 1, 8, 27, etc., formed? What is the meaning of 2^3? Fill up the table until your first number reads '55'.

c Look at the following:

$$\begin{array}{lr}
1^3=\ 1 & 1=\ 1^2\\
2^3=\ 8 & 1+\ 8=\ \ 9=\ 3^2\\
3^3=27 & 1+\ 8+27=\ 36=\ 6^2\\
4^3=64 & 1+8+27+64=100=10^2
\end{array}$$

Find the patterns in this table. Where are the triangular numbers? Continue the tables up to the line beginning 10^3.

Exercise 2 **a** Which of the triangular numbers you have used are primes?

b Write down the sets of divisors of the 4th, 6th, and 8th triangular numbers.

c Write down the sets of divisors of 25 and 49.

d The factors of a number are divisors which when multiplied together give the number. Write down pairs of factors for (i) 6, (ii) 12, (iii) 16. (There are for (i) two pairs, for (ii) three pairs, for (iii) three pairs.)

e Sometimes a pair of factors consists of two equal numbers: $25 = 5 \times 5$. Say which of the following have a pair of factors like this:
12, 9, 14, 16, 1, 26, 49, 28, 0, 36.

f Write down each of the answers to (e) in a different way.

Now $5 \times 5 = 5^2 = 25$. We say that '5 is the square root of 25'. In symbols we write: $5 = \sqrt{25}$. The sign $\sqrt{\ }$ is read 'is the square root of', or, for short, 'root'.

Exercise 3 **a** Write down in another way $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, $\sqrt{36}$, $\sqrt{49}$.

b Work out the following:

i $\sqrt{9}+\sqrt{16}$

ii $\sqrt{36}+\sqrt{49}$

iii $\dfrac{\sqrt{36}}{\sqrt{1}+\sqrt{9}}$

iv $\dfrac{\sqrt{49}}{\sqrt{1}+\sqrt{4}+\sqrt{9}}$

v $\dfrac{\sqrt{4}+\sqrt{36}}{\sqrt{64}}$

vi $\sqrt{\sqrt{100}+\sqrt{36}}$

vii $\dfrac{\sqrt{256}}{\sqrt{9}-\sqrt{4}}$

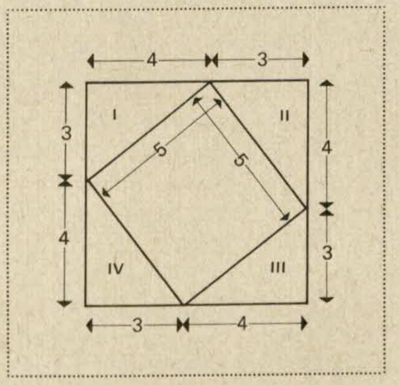
c Which numbers between 100 and 121 do not have a whole number square root?

Exercise 4 **a** i What is $9+16$?
ii What is 3^2+4^2
iii $5^2-4^2 = \quad ?$
iv What can you say about the numbers 3, 4, 5?

b Is the same kind of thing true of 6, 8, 10 and 9, 12, 15?

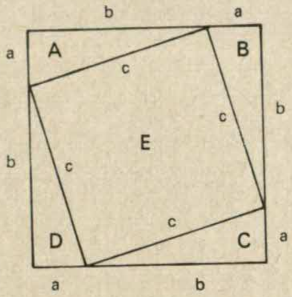
c What is the connexion between the two sets of numbers in (b) and 3, 4, 5?

d



Write down the following areas:
i square V
ii triangle I
iii triangle I + triangle II + triangle III + triangle IV
iv the large square
What do you notice about the results (i), (iii) and (iv)?

e



What is

i the length of a side of the large square?

ii the area of the large square?

iii the sum of the areas A, B, C, D, E?

f In the last diagram $(a+b)^2 = c^2 + 4\,(\frac{1}{2}ab)$. Why?

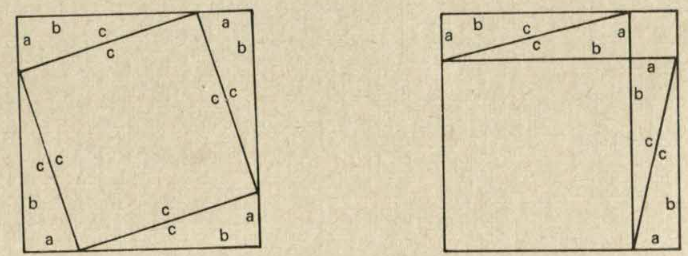
g Remove the brackets from the left hand side and simplify $4(\frac{1}{2}ab)$. Rewrite the whole statement in its new simple form.

h You should have written down

$a^2+b^2+2ab = c^2+2ab$

in the last question. If you subtract $2ab$ from both sides, what do you get?

i

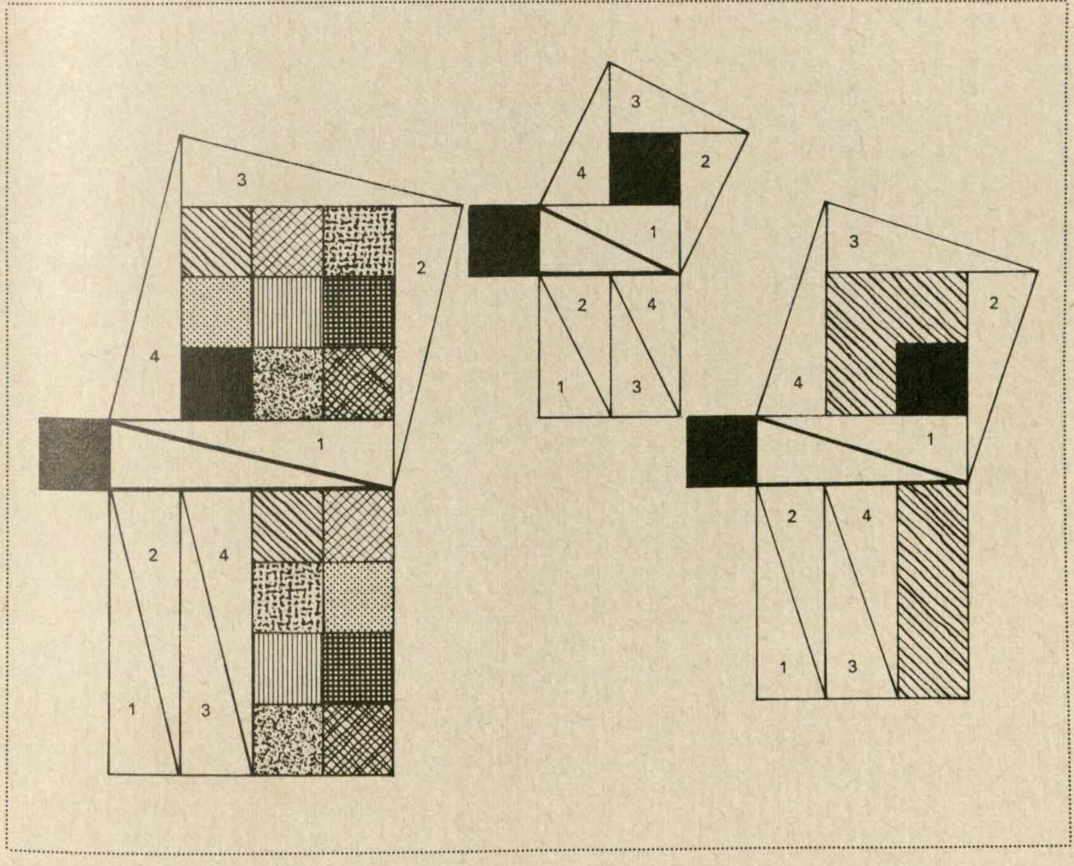


A

B

Copy diagram (A) on to drawing paper, making the large square $2\frac{1}{2}$ in. each way. You may make a and b any convenient size. Letter your diagram as we have done. Also draw a $2\frac{1}{2}$ in. square and leave it blank. Cut out your first diagram and cut off the four triangles. Can the triangles be arranged on the blank square as in diagram (B)? You now have three squares. One cut from diagram (A) and two on the blank square. Write inside each its area. What does this tell you about a^2, b^2, and c^2? Look at one of your triangles. What sort of triangle is it? What are a, b, and c? Could you cut out the triangles from diagram (B) and fit them on diagram (A)?

You have now discovered for yourself a fact which was known for many thousands of years. It was probably used by the men who built the pyramids. It was also well known to the Greeks. It used to be said that a man called Pythagoras first proved the fact round about 550 BC. It is called 'Pythagoras' theorem' and relates to right angled triangles only. It states that 'the square on the longest side (the hypotenuse) of a right angled triangle is equal to the sum of the squares on the other two sides.'



Exercise 5 The diagrams show in several ways that the sum of the two smallest square areas is equal to the largest. They are most easily drawn on squared paper. See if you can find some other ways of showing the fact.

Remember, in a right-angled triangle whose sides are a, b, c and $c > a$ and $c > b$, we have $a^2 + b^2 = c^2$.

Exercise 6 You have already seen that the set $\{3, 4, 5\}$ is a solution set of $\{(a, b, c) | a^2+b^2 = c^2 ; a, b, c \text{ are integers.}\}$

a Is it true that $\{3k, 4k, 5k\}$ is another solution where k is *any integer*?

b Say which of the following are also solution sets:

i $\{5, 12, 13\}$
ii $\{6, 9, 12\}$
iii $\{7, 15, 16\}$
iv $\{7, 24, 25\}$
v $\{8, 15, 17\}$
vi $\{5t, 12t, 13t\}$

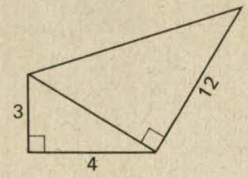
(t is any integer).

Exercise 7 Place numbers in the frames to make true statements of the following open statements. (Remember what you discovered in Exercise **6a** and **b**[vi].)

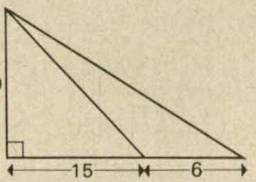
a $3^2+4^2 = \square^2$
b $\triangle^2+24^2 = 25^2$
c $5^2+\nabla^2 = 13^2$
d $11^2+\square^2 = 61^2$
e $21^2+72^2 = \nabla^2$
f $4^2+\triangle^2 = 5^2$
g $\square^2+120^2 = 123^2$
h $10^2+24^2 = \square^2$
i $\triangle^2+12^2 = 20^2$
j $15^2+\nabla^2 = 17^2$
k $16^2+30^2 = \square^2$
l $\nabla^2+21^2 = 29^2$
m $119^2+\nabla^2 = 169^2$
n $\triangle^2+40^2 = 41^2$
o $27^2+\square^2 = 45^2$
p $\triangle^2+64^2 = 136^2$

Exercise 8 Find the lengths of the unmarked lines.

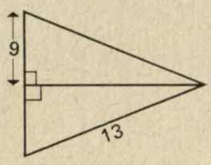
a



c



b



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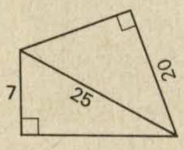


Answers

Exercise 7 a 5 c 12 e 75 g 27 i 16 k 34 m 120 o 36

Exercise 8 a 5, 13 c 25, 29

Section 22 **Mainly square roots**

Exercise 1 **a** Write down the values of the following:

i $\sqrt{169}$

ii $\sqrt{a^2}$

iii $\sqrt{b^{10}}$

iv $\sqrt{4b^2}$

v $\sqrt{16 \times 144}$

vi $\sqrt{(c^4)^2}$

vii $\sqrt{5 \times 45}$

viii $\sqrt{0{\cdot}04}$

ix $\sqrt{0{\cdot}0001}$

x $\sqrt{\frac{1}{16}}$

xi $\sqrt{\frac{4}{25}}$

xii $\sqrt{\dfrac{3 \times 24 \times 32}{7 \times 28}}$

b i If a certain square is known to have an area of 400 sq. in., how long must the sides be?

ii I intend to retile part of a drawing room floor. The room is 12 ft. wide and 14 ft. long. I have a square piece of carpet just under $13\frac{1}{2}$ sq. yds. in area. On measuring the carpet I find that it is a whole number of square feet. I intend to retile only that part of the floor which surrounds the carpet, plus a single row of tiles under the outside edge of the carpet. If the tiles are each 1 ft. square, find out how many new tiles I shall need. Draw a scale diagram to show how I can arrange the floor with the carpet as near to the middle of the floor as possible.

Exercise 2 **a** Find the cube roots of the following:

i 8

ii 27

iii 1728

iv a^3

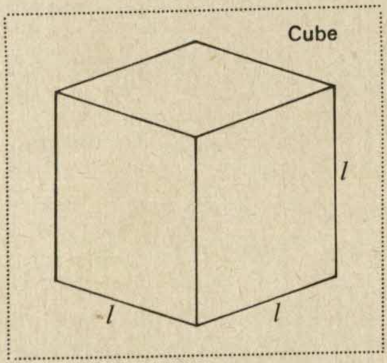
v b^6

vi $(c^5)^3$

vii d^{33}

viii $1{,}000\, e^{18}$

b



A box in the shape of a cube will just hold 2197 c.c. of fine powder. How long are the edges of the cube (inside measurements)? If the box is $\frac{1}{8}$ in. thick and has a lid of the same thickness, what are the outside measurements?

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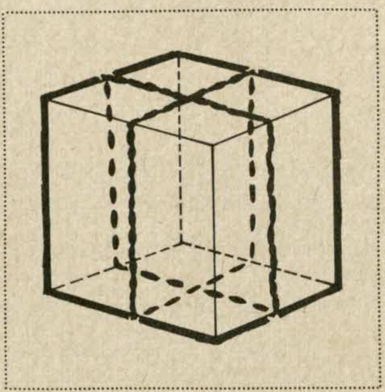
c I have just bought a quantity of sand for making mortar. The man who delivered it said it was two yards of sand. What did he mean? I am storing it in an old bunker which is very nearly cubical in shape, being of square base but slightly less tall than it is wide. It just holds the sand. About how long and wide is the bunker inside? (You will need to consider whether the length is nearer to 1 yd. or 2 yds. and then to try some values. You can easily get the length to the nearest $\frac{1}{8}$ yd.)

d 

Here is a puzzle about a parcel I am packing. The parcel is made up of a number of recording tapes in boxes. I have used 64 in. of string to tie it, of which 4 in. was for the knots. The parcel is a cube. Each box is $\frac{1}{11}$ times as deep as it is wide, and is as long and wide as the parcel. What is the volume of the parcel and how many boxes are there?

Exercise 3 **a** What is a prime number?

b What do we mean by factors?

c What do we mean by prime factors?

d How can you often find the square root of a number by factorizing it? Make up an example to show what you mean.

e Make up an example to show how you could also find a cube root by factorizing the number.

Exercise 4 **a** Factorize the number 462.

b Write down the square of 20.

c One of the two following statements is true and the other false. Which is which?

i $\sqrt{400} > \sqrt{462}$ ii $\sqrt{400} < \sqrt{462}$

d Is it true to say that $20 < \sqrt{462} < 22$? Explain.

Exercise 5 A perfect square is an integer which is the exact square of another integer. In this work we are only concerned with positive integers.

a Find the square root of the nearest perfect square less than each of the following integers:

i 105 iii 60 v 1074
ii 154 iv 91 vi 6865.

b Factorize each of the numbers in (a).

c Write down the square root of the nearest perfect square greater than each of the numbers in (a).

d For each of the numbers in (a) write down a statement like this which is for (i): $10 < \sqrt{105} < 11$

e From what you have just done, do you think that you can *always* write down limits between which a square root must lie?

f Examine your prime factors again. Do you think that when you are trying to factorize a number it is ever necessary to try division by any prime bigger than the square root of the number? Explain your answer carefully and give two examples which you have made up.

Exercise 6 **a** Complete each of the following statements. The sign $\doteqdot$ means *approximates to.*

i $63{\cdot}25^2 =$. iii $632{\cdot}5^2 =$.
ii $\sqrt{4000} \doteqdot$. iv $\sqrt{400{,}000} \doteqdot$.

b Complete the following by putting in the exact value or, where this is not possible, altering $=$ to $\doteqdot$ and giving an approximate value.

i $\sqrt{4} =$. iv $\sqrt{4000} =$.
ii $\sqrt{40} =$. v $\sqrt{40{,}000} =$.
iii $\sqrt{400} =$. vi $\sqrt{400{,}000} =$.

c Write out the first ten terms of the set $\{x|x = 4 \times 10^n; n \text{ is an integer} \geqslant 0\}$.

d Find a partition of the set you have just written into two subsets. The square roots of the elements in each subset are to be similar in some way.

Explain what the similarity is and how you may continue to select the elements without writing them down.

e Call your subset containing 4, A, and the other B. Say which subset, A or B, the following elements belong to:

i The 15th element of the original set.
ii The square root of 4 followed by 12 zeros.
iii 4×10^{57}.

f Write out the first three terms of each of the following sets and in each case show how they can be separated into subsets A and B as before.

i $\{x \mid x = 2 \times 10^n;\ n \text{ is an integer} > 0\}$
ii $\{x \mid x = 7 \times 10^n;\ n \text{ is an integer} > 0\}$
iii $\{x \mid x = 1 \times 10^n;\ n \text{ is an integer} > 0\}$

When mathematicians want square roots we usually use a table to find them. The work you have just done makes two things clear; one is that we need two tables for any given set of numbers corresponding to A and B in the exercise; the other is that provided we can ignore the decimal point, there is only need to give one square root for the numbers in a given subset. For instance if we want to list the square roots of all the numbers like 4, 40, 400, 4000 and so on, we need only include in the table 2 and 6325. This means however that we have to estimate the position of the decimal point ourselves. There is a table of square roots at the end of the Section. The values are not exact except in the case of the perfect squares. What has been done to the other values?

Exercise 7 **a** What is the nearest perfect square below 150? What is its square root?

b What is the nearest perfect square above 150? What is its square root?

c $12 < \sqrt{150} < 13$. Find 15 in the table of square roots. Of the two numbers in the square root column can you find one which will give a value between 12 and 13 if you place a decimal point in the appropriate place?

d Go through the same line of reasoning with 15.

e Use the table to find the approximate square roots of the following:

i 10 iii 33 v 4,300 vii 40
ii 4·9 iv 590 vi 2·8 viii 1·5

f What is the square of each of the following?

i 0·2 iii 0·1 v 0·06 vii 0·004
ii 0·7 iv 0·03 vi 0·01

g What is the square root of each of the following?

i 0·0001 iii 0·01 v 0·000016 vii 0·0036
ii 0·04 iv 0·49 vi 0·0009

h How many digits are there in each of the numbers of (g)? Were there any odd numbers of digits?

i Find the squares of the following numbers:

i 0·3 ii 0·4 iii 0·7 iv 0·8

j Do you think that the following two statements are true or false?

i $0{\cdot}3 < \sqrt{0{\cdot}1} < 0{\cdot}4$ ii $0{\cdot}7 < \sqrt{0{\cdot}5} < 0{\cdot}8$

k Use the table to find the approximate square root of

i 0·1 ii 0·5

l Think about the answers to the last question. Is 0·1 the same as 0·10? Is 0·5 the same as 0·50? Is 0·3 the same as 0·30? What are the square roots of 10, 50, and 30?

m Can you now see how to find the square roots of 0·2, 0·4, 0·6? Use your table to find them.

n Use the knowledge you now have to find the approximate square roots from the table of the following numbers. Check back by rough multiplication that you have placed the decimal point in the right position.

i 0·00005 ii 0·003 iii 0·002 iv 0·00004

o Find the square roots of the following using the table where necessary. Two examples are given. Check your answers by rough calculation.

$\sqrt{4 \times 10^2} = \sqrt{4} \times \sqrt{10^2} = 2 \times 10 = 20;$

$\sqrt{25 \times 10} = \sqrt{2{\cdot}5 \times 10^2} = \sqrt{2{\cdot}5} \times \sqrt{10^2} \doteq 1{\cdot}58 \times 10 = 15{\cdot}8$

i 4×10^4
ii 9×10^6
iii 16×10
iv 121×10^4
v 41×10^6
vi $3{\cdot}4 \times 10^5$
vii 4×10^3
viii 52×10^4
ix $5{\cdot}2 \times 10^3$
x $5{\cdot}2 \times 10^5$
xi $5{\cdot}2 \times 10^4$
xii $0{\cdot}69 \times 10^4$
xiii $0{\cdot}81 \times 10^3$
xiv $25 \div 10^4$
xv $2{\cdot}5 \div 10^3$
xvi $79 \div 10^4$

Exercise 8 **a** Find $\sqrt{36}$ and $\sqrt{64}$. Now find $\sqrt{(36+64)}$, and $\sqrt{36}+\sqrt{64}$.

b Find $\sqrt{54}$ and $\sqrt{23}$. Now find $\sqrt{(54+23)}$ and $\sqrt{54}+\sqrt{23}$.

c Find $\sqrt{0{\cdot}45}$ and $\sqrt{0{\cdot}57}$. Is $\sqrt{(0{\cdot}45+0{\cdot}57)} = \sqrt{0{\cdot}45}+\sqrt{0{\cdot}57}$?

d Do you think that

$$\{x, y \mid \sqrt{x}+\sqrt{y} = \sqrt{(x+y)};\ x, y \text{ are real } > 0\}$$

is a null set?

e Is there a solution set of

$\{x, y \mid \sqrt{x}+\sqrt{y} = \sqrt{(x+y)};\ x, y \text{ are real}\}$

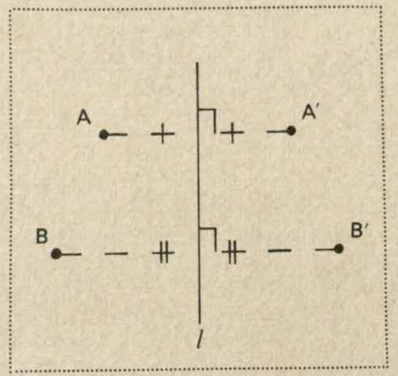
If so how many elements are there in it?

A table of square roots from 10 to 99

No.	square root	No.	square root	No.	square root	No.	square root
10	100 316	33	182 575	56	237 748	79	281 889
11	105 331	34	184 583	57	239 755	80	283 894
12	110 346	35	187 592	58	241 762	81	285 900
13	114 361	36	190 600	59	243 768	82	286 906
14	118 374	37	192 608	60	245 775	83	288 911
15	123 387	38	195 616	61	247 781	84	290 917
16	127 400	39	198 625	62	249 787	85	292 922
17	130 412	40	200 633	63	251 794	86	293 927
18	134 424	41	203 640	64	253 800	87	295 933
19	138 436	42	205 648	65	255 806	88	297 938
20	141 147	43	207 656	66	257 812	89	298 943
21	145 458	44	210 663	67	259 819	90	300 949
22	148 469	45	212 671	68	261 825	91	301 954
23	152 480	46	215 678	69	263 830	92	303 959
24	155 490	47	217 686	70	265 837	93	305 964
25	158 500	48	219 693	71	267 843	94	307 970
26	161 510	49	221 700	72	268 848	95	308 975
27	164 520	50	224 707	73	270 854	96	310 980
28	167 529	51	226 714	74	272 860	97	311 985
29	170 539	52	228 721	75	274 866	98	313 990
30	173 548	53	230 728	76	276 871	99	315 995
31	176 557	54	232 735	77	277 878		
32	179 566	55	235 742	78	279 883		

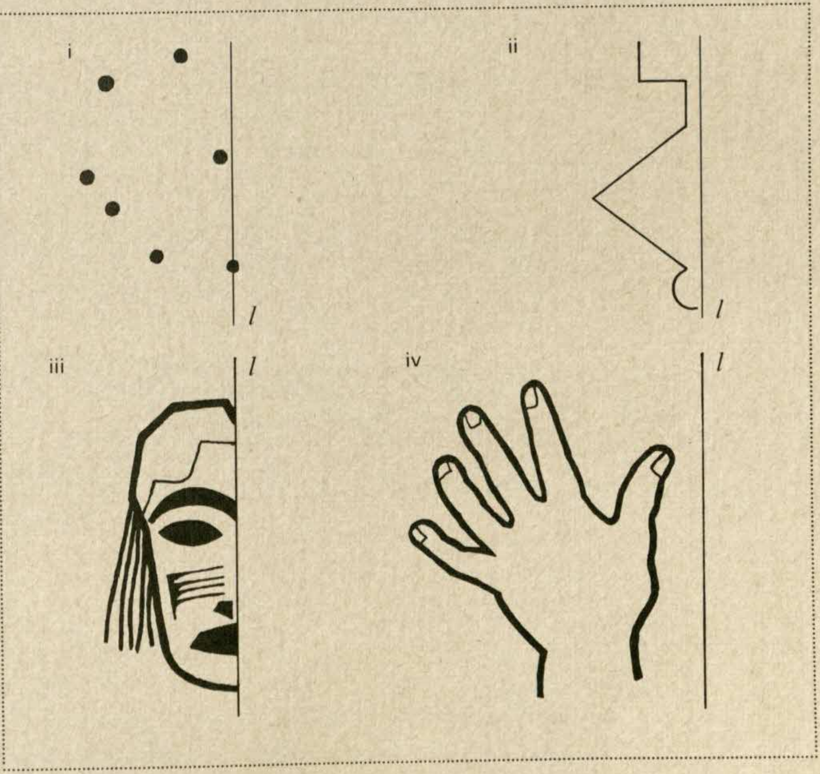
Section 23 **Reflection**

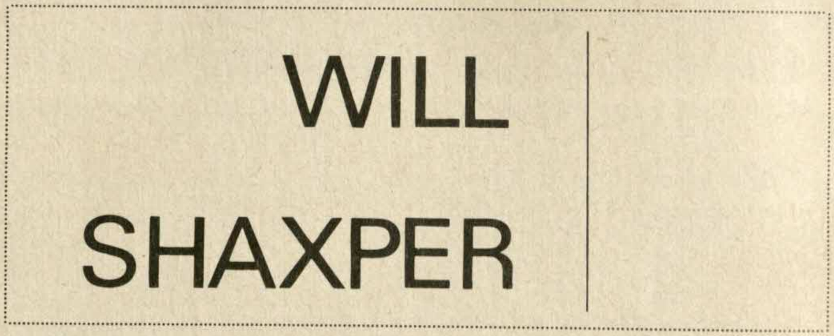
Exercise 1 **a**



Look at this diagram. The axis *l* is an axis of reflection. The diagram shows how we can reflect A and B in the axis so that their images are A′ and B′. Could we have just as easily begun with A′ and B′ and reflected them so that their images are A and B?

b



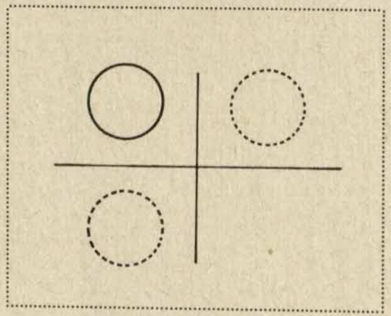


(*an old spelling*)

Use squared paper to copy each of the above drawings approximately and then to draw their images.

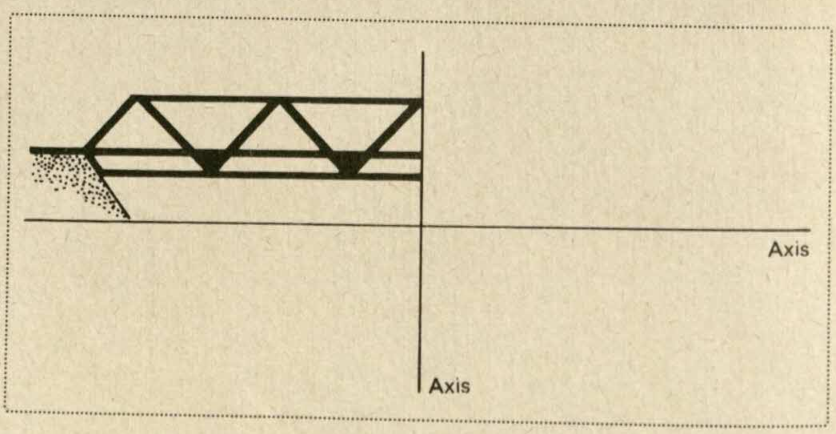
c Which letters of the alphabet reflect into a shape identical with the original?

d



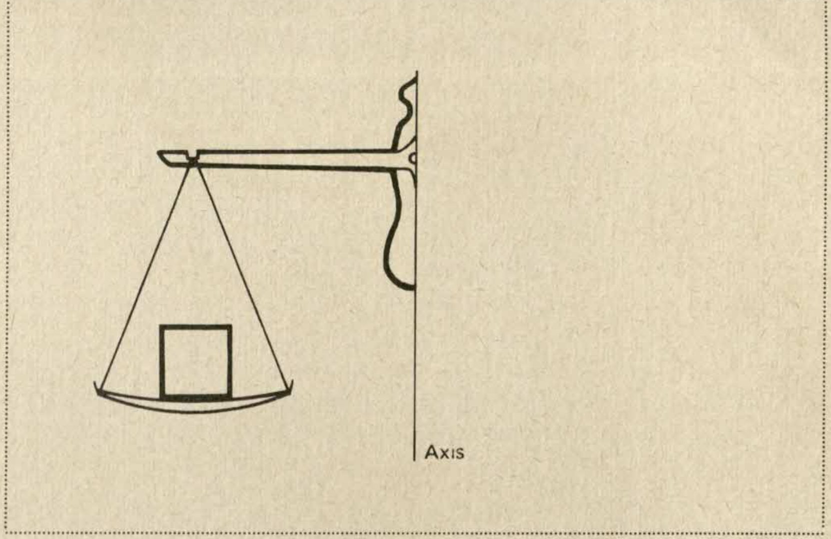
Can you find which letters have the same image whichever of the two axes of reflection we use?

e



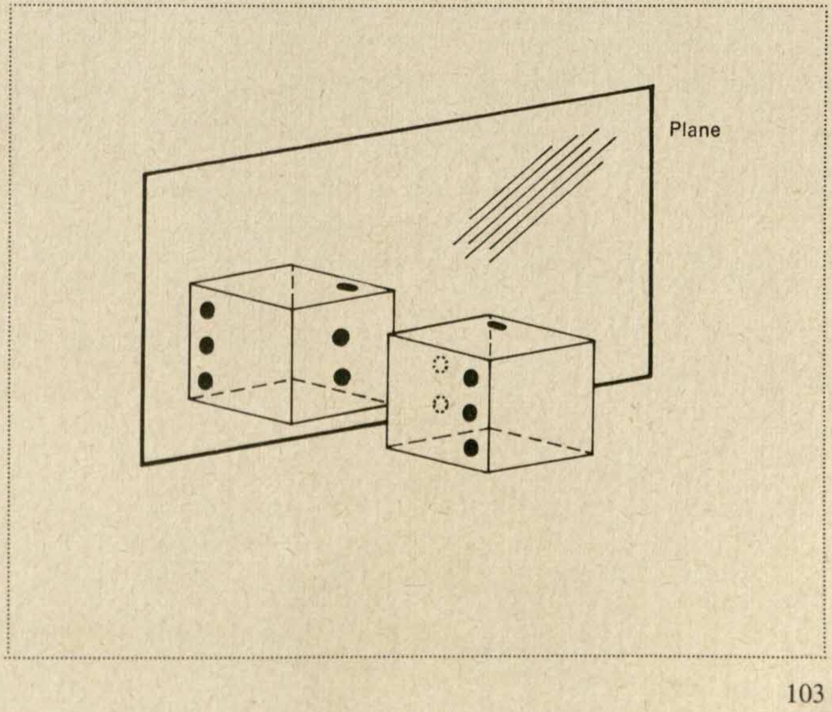
Here is part of a symmetrical bridge. Carry out the two reflections which will complete the drawing of the bridge and its reflection in the water.

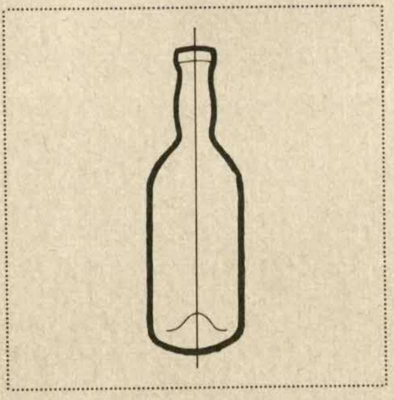
f



Carry out reflections in the axis to complete this picture of an apothecary's balance.

Exercise 2 Here is a rectangular block reflected in a plane.





Here is a wine bottle formed by a reflection in a plane of symmetry.

a Make a list of four common objects which have a plane of symmetry and of three more whose images in planes outside them are identical to the objects.

b Is a wine bottle (with its label on) symmetrical about a plane?

c Is a left-hand glove reflected into a left-hand glove?

d Is a man identical to his reflection in a mirror? What happens to him when he is reflected?

e Do you know your own face? Why does a photograph of you look strange to you?

Exercise 3 **a** Which plane figures with

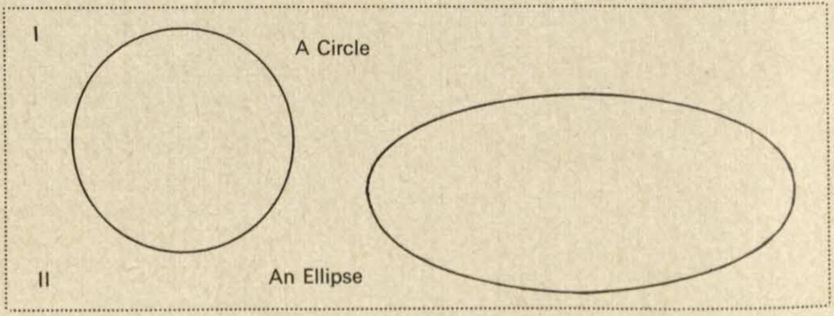
i 3 straight sides

ii 4 straight sides

iii 5 straight sides

are symmetrical about a line passing through them? How many such lines are there in each case?

b



How many axes of symmetry are there for:

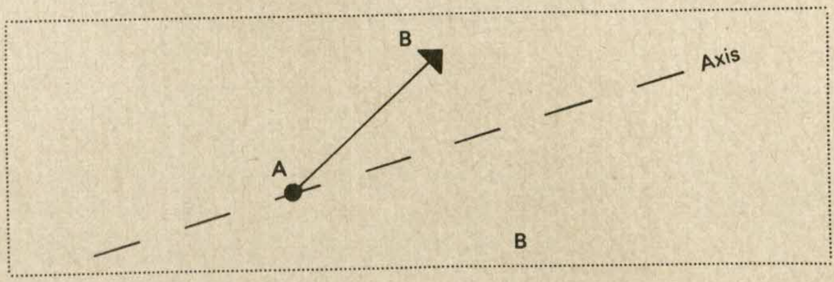
i a circle?

ii an ellipse?

c How many planes of symmetry pass through

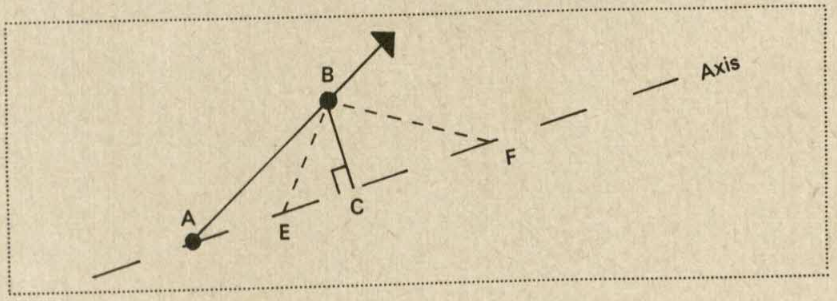
i a cone
ii a rectangular block
iii a right pyramid on a square base
iv a right pyramid on a base which is an equilateral triangle
v a parabolic reflector
vi a sphere.

Exercise 4 **a**



Complete the following reflection:
The axis is then said to be the *bisector* of the acute angle BAB′.

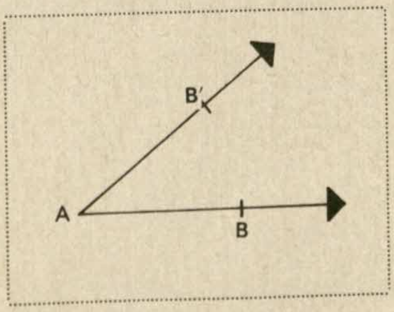
b



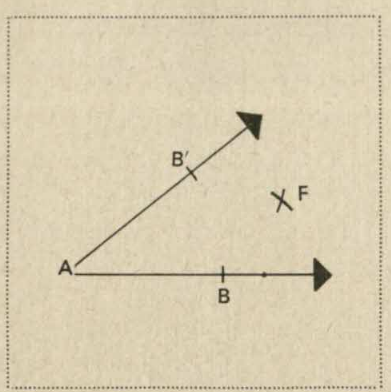
Here is a more elaborate version of the same diagram.

i Complete the reflection and mark equal angles and equal segments.
ii What are the images of A, E, F?
iii What do you know about any point lying, like E and F, on the axis?

c



Draw an acute angle in the centre of your page.
Now with your compasses centre A mark B and B′.



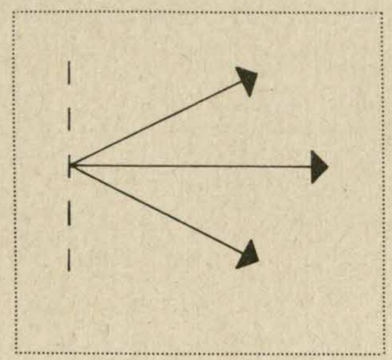
Now with centres B and B′ and any radius (the same for each) draw two arcs to intersect. Call this point F. Join AF, and produce in both directions.

d Explain what the construction you have done has to do with the diagram in (b).

e Could you have used the radius AB for BF and B′F?

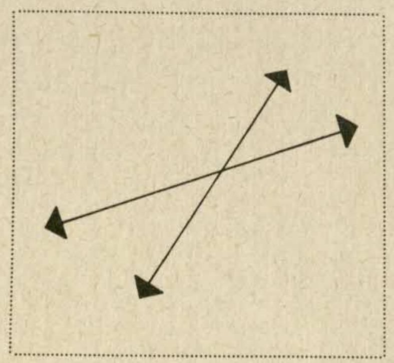
f Could you have used a radius in such a way as to give a point like E in (b)? Would it have been wise?

g



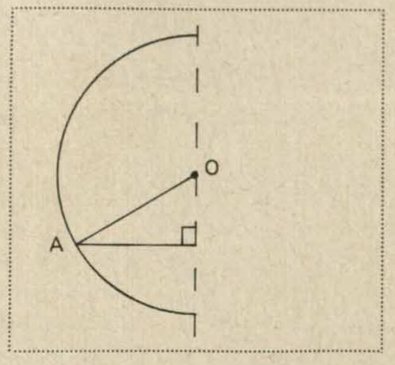
Here is a pair of half lines and the bisector of the angle between them. The axis dotted in is an axis of reflection at right angles to the bisector.

- i Complete the reflection
- ii Where are there right angles?
- iii Where are there vertically opposite angles?
- iv Which lines are straight?



- v How many bisectors are there for the angles between a pair of lines?

Exercise 5 **a**

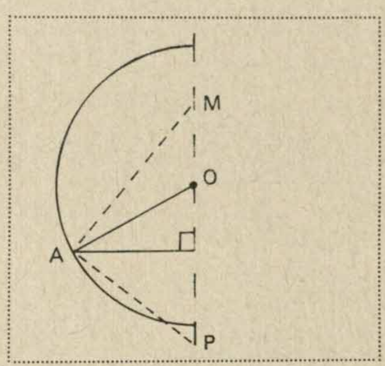


Here is a semi-circle with centre O and an axis of reflection.

i Complete the reflection

ii What sort of triangle have you drawn?

b



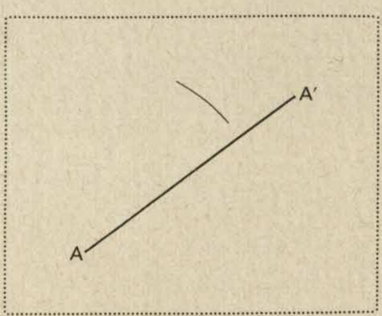
Here is a more complicated figure, also with a semicircle whose centre is at O.

i Complete the reflection

ii What can you say about AM and A′M? about AP and A′P?

iii What can you say about the lines MP and AA′?

c



i Copy the line-segment AA′. With your compasses on A and a radius bigger than half the length of AA′, draw an arc as we have done. Now use the same radius and A′ as the centre and draw an arc to intersect the first. Now choose another radius also bigger than half the length of AA′ and draw two arcs to intersect below AA′. (Use A and A′ as your centres again.) Call the points of intersection respectively M and P. Join MP, and produce in both directions.

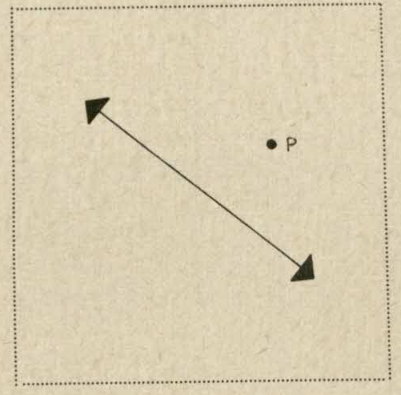
ii You have drawn the *right-bisector* of AA′. Why do you think it is given this name?

iii Explain what your construction has to do with the reflection in (b) above.

iv Could you have used the same radius for all four arcs?

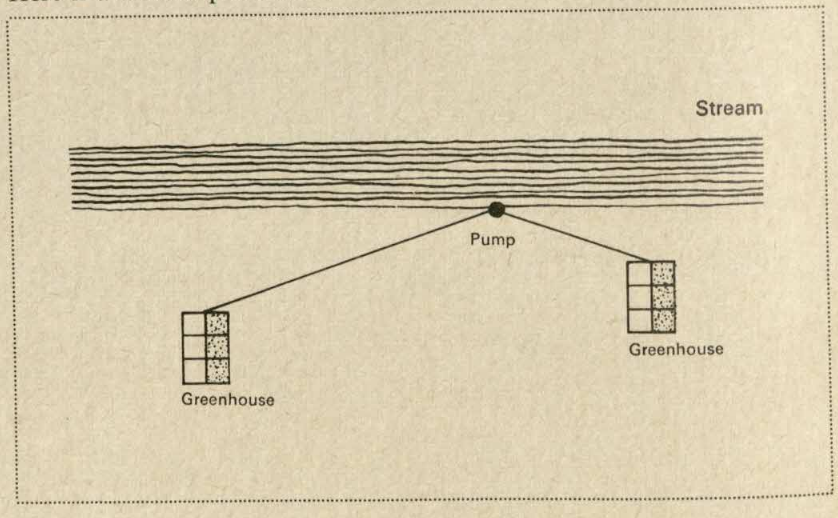
v Need the second pair of arcs be 'below' the line? Explain.

d



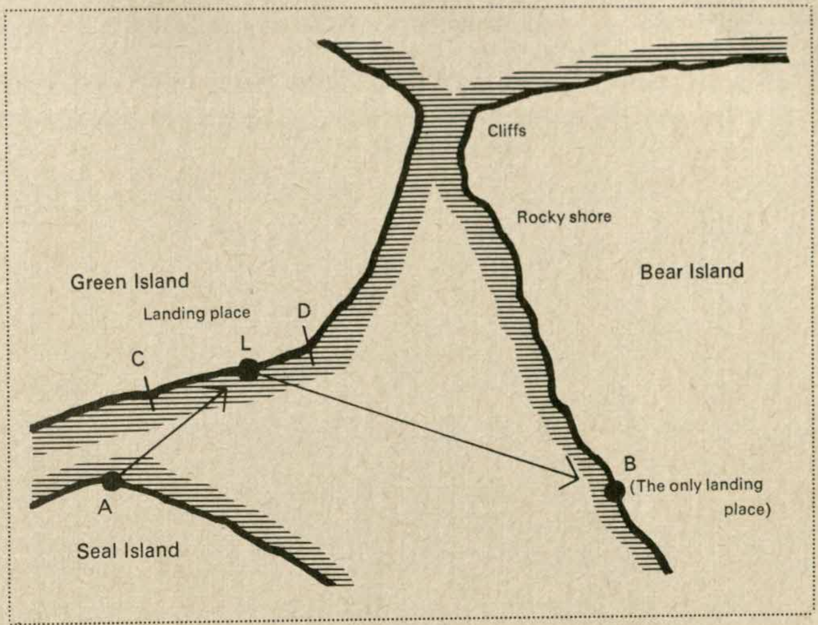
Use what you know to invent a construction for drawing a line from a point P at right angles on to a line *l*.

Exercise 6 **a** Here is a famous problem:



From the stream it is intended to draw water for each of the two greenhouses. A pump will be installed at a convenient point on the bank. In order to save money it will obviously be helpful if the pipes from the pump P to the greenhouses G_1 and G_2 are as short as possible. Where should the pump be located?

b

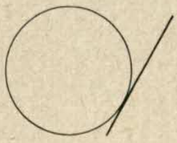


A group of explorers is on an expedition to an archipelago. They have a base on the only landing place on the south-west side of Bear Island and they have just completed the exploration of Seal Island. They are anxious to have a look at Green Island before they return to base on Bear Island. They can see that the coast line between two points (C and D on the map) is more or less straight and has a shelving beach quite suitable for landing. They are rather short of fuel for the launch and want to make the journey from A to L to B as short as possible. Where should they land on Green Island? Make a tracing of the map and find the exact position of L by drawing.

c The straight line and the circle are in contact at one point only.

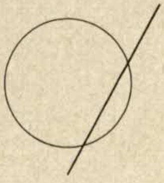
i How would you describe the straight line?

ii Reflect the whole diagram in the straight line as axis. In how many points do the circles intersect?



iii What can you say about the line joining the centres of the two circles in the finished diagram?

d The straight line and the circle intersect in two points. Repeat **c**(i), (ii) and (iii) for this diagram.



Something more about primes

1	2	3	$\underline{4}$	5	$\underline{\underline{6}}$	7	$\underline{8}$	$\underline{9}$	$\underline{\underline{10}}$
11	$\underline{\underline{12}}$	13	$\underline{\underline{14}}$	$\underline{\underline{15}}$	$\underline{16}$	17	$\underline{\underline{18}}$	19	$\underline{\underline{20}}$
$\underline{\underline{21}}$	$\underline{22}$	23	$\underline{\underline{24}}$	$\underline{25}$	$\underline{26}$	$\underline{27}$	$\underline{\underline{28}}$	29	$\underline{\underline{\underline{30}}}$
31	$\underline{32}$	$\underline{33}$	$\underline{34}$	$\underline{\underline{35}}$	$\underline{\underline{36}}$	37	$\underline{38}$	$\underline{39}$	$\underline{\underline{40}}$
41	$\underline{\underline{\underline{42}}}$	43	$\underline{44}$	……………………etc.					
$\underline{51}$	………………………………etc.								

etc.

In the table we have begun, certain numbers are underlined. This has been done this way: first we underlined the multiples of 2 (except for 2 itself) with just one line. Next we underlined the multiples of 3 (except for 3 itself); since some of these were already underlined, we underlined again to prevent our making errors. We then dealt with the multiples of 5 which was the next number not already underlined, but we did not underline the first multiple of 5, i.e. 5. The next number not underlined was 7, so we carried on the same way.

Exercise 1 **a** Complete the table to 200 in your book and very neatly carry on underlining the numbers according to the method we have begun. Do the first part without looking at the table above and then check it.

b What sort of numbers are left not underlined at all?

c What is the meaning of the fact that numbers have sometimes been underlined several times? What can you tell from the number of underlinings of a number (even if that number of underlinings is 0)?

d You have constructed a SIEVE OF ERATOSTHENES. This is the only way you can be sure of obtaining all the primes in sequence. You could carry on as long as you wish, but clearly, the work becomes very laborious after a time. See what you can find out about Eratosthenes and the other work he did in mathematics.

e We are going to introduce a new symbol for this question. You may have met the sign before, but the meaning here is something quite different. $\Pi(n)$ will stand for the number of primes below and including a number n. For instance, $\Pi(12) = 5$; $\Pi(17) = 7$. By looking at your sieve, find

i $\Pi(21)$ iii $\Pi(90)$ v $\Pi(200)$
ii $\Pi(37)$ iv $\Pi(100)$

You can make up some interesting numbers from certain formulae which have been invented. They were discovered when men tried to find a rule which would give them a method of working out all the prime numbers. It can now be proved that no such rule can exist. Some rules have however been found which give some of them.

Exercise 2 $M_r = 2^r - 1$, where r is a natural number >0.

a $M_1 = 2^1 - 1 = 1$; $M_2 = 2^2 - 1 = 3$. Find M_3, M_5, M_7.

b These values of M_r are not primes: M_4, M_6, M_8. Work them out and find their factors.

It has been shown that M_r is prime when

$r = 1, 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127$.

In fact M_{127} has been worked out as

170, 141, 183, 460, 231, 731, 687, 303, 715, 884, 105, 727.

All those numbers from M_{128} to M_{251} have been calculated and found to be composite, that is, not prime.
The formula $M_r = 2^r - 1$ was discovered by Mersenne in 1644.

Exercise 3 $F_t = 2^{2^t} + 1$ is another formula. Work out F_t for

a $t = 0$ **b** $t = 1$ **c** $t = 2$ **d** $t = 3$
All of these are prime.

F_4 is 65,537 but F_5 turns out to be a multiple of 641. These numbers F_t are the famous Fermat numbers.

Section 25 **More algebra**

We have already found (p. 81) an expression for $(a+b)\times(c+d)$ or, leaving out the multiplication sign, $(a+b)(c+d)$.

We found that $(a+b)(c+d) = ac+bd+ad+bc$ and we did it by thinking about the areas of rectangles. You can remember this result either by the pattern $(a+b)\ (c+d)$ or by the pattern $(a+b)\,(c+d)$ which is the same thing. In words the second one is: multiply the second bracket by a and then by b. The letters a, b, c, d stand for any real numbers you care to choose.

Suppose you chose $a = 2$, $b = 3$, $c = 4$, $d = 5$. The left-hand side (LHS) of the expression is then $(2+3)(4+5) = 5\times9 = 45$. The right-hand side (RHS) is $2\times4+3\times5+2\times5+3\times4 = 45$. Thus for these numbers the statement certainly holds true.

Exercise 1 Check the LHS and RHS of the expression for each of the sets of values given.

a $a = 1,\ b = 2,\ c = 3,\ d = 4.$

b $a = 4,\ b = 5,\ c = 2,\ d = 2.$

c $a = 9,\ b = 1\frac{1}{2},\ c = 7,\ d = 8.$

d $a = 2,\ b = 0,\ c = 3,\ d = 4.$

e $a = 2,\ b = 3,\ c = 0,\ d = 1.$

f A set of values of your own choice.

Exercise 2 **a** Does the expression still work when one of the quantities is zero?

b What would be the result of each side if $b = 0$ and the others are left as letters?

c Suppose a and b were equal; would the expression then be correct? Let $a = b = x$. What does the expression now become on each side?

d Suppose $a = c$ and $b = d$. What is the expression now on both sides?

Exercise 3 You have now a method of removing the brackets from $(a+b)^2$. What you are really saying is that when you see this expression you can replace it by (a^2+b^2+2ab) or by $(a^2+2ab+b^2)$ which are equivalent expressions. (Reminder: look back to p. 83.)

a $1{\cdot}01 = 1+0{\cdot}01$. So $(1{\cdot}01)^2 = (1+0{\cdot}01)^2$.
Use the expression you have discovered to work out the value of this. Work this out the usual way as a check.

b Use the same method to simplify each of the following. Do each again by the more usual method.

i 57^2
ii 121^2
iii 1009^2
iv 49^2
v $2{\cdot}8^2$

c Use your method to write each of the following in a different form:

i $(e+f)^2$
ii $(5+g)^2$
iii $(x+6)^2$
iv $(2x+1)^2$
v $(4x+5)^2$
vi $(3y+z)^2$
vii $(6x+7z)^2$
viii $(3c+4d)^2$
ix $(x^2+1)^2$
x $(3y^2+8)^2$
xi $(6v^2+5w)^2$
xii $(7f^2+8g^2)^2$
xiii $(h^3+j^4)^2$
xiv $(9m^5+7n^6)^2$

d From what you have learned try to write each of the following in brackets.

i x^2+2x+1
ii y^2+6y+9
iii $z^2+16z+64$
iv $a^2+10a+25$
v $b^2+14b+49$

e Make up five problems like the ones you have just done in (d) and when you are sure you know exactly how to do them exchange your problems with your neighbour and solve his or hers.

f Say which of the following statements are true and which are false. Use what you have just learned.

i $3^2+4^2 = 7^2$
ii $3^2+4^2 = 5^2$
iii $1^2+1^2 = 2^2$
iv $1^2+1^2 = 1+1+2$
v $(1+1)^2 = 1^2+1^2+2$
vi $(3+4)^2 = 3^2+4^2$
vii $(3+4)^2 = 3^2+4^2+24$
viii $(4^2+5^2)^2 = 16^2+82+625$

Exercise 4 **a** i Work out (i.e. *expand*) $a(x+y)$
ii In the last expression, let $a = x+y$. What is the expansion now? Simplify it as much as you can.
iii In the same expression, let $a = v+w$. Simplify the result.

b Expand

i $(x+2y)(2x+y)$ ii $(x+2y)(x+2y)$ iii $(2x+3y)^2$

c i Complete the following statement: $pq+pr = p(\quad)$.
ii Form a similar expression from $2fx+2gx$.
iii Do the same with $3mx+4my$ and $6ax+8bx$.

d i Complete the following statement: $(x+y)r+(x+y)s = (x+y)(\quad)$.
ii Form similar statements from $(x+y)a+(x+y)b$ and $(x+2y)p+(x+2y)q$.
iii Repeat with $2(a+b)x+2(a+b)y$ and $2(a+b)m+3(a+b)n$.
iv Repeat with $4(p+q)r+6(p+q)s$ and $6(x+y)y+8(x+y)z$.

e Complete the following statements:

i $x^2+xy+xy+y^2 = x(x+y)+y(\quad) = (x+y)(\quad)$
ii $x^2+2xy+2xy+4y^2 = x(\quad)+y(\quad) = (x+y)(\quad)$
iii $4x^2+6xy+6xy+9y^2 = (2x+3y)+3y(\quad) = (\quad)(\quad)$
iv $x^2+3xy+3xy+9y^2 = (\quad)+(\quad) = (\quad)(\quad)$

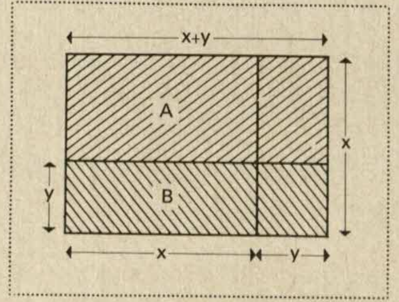
Exercise 5 **a** Say whether the following are true or false:

i $2(4+5) = 24+25$
ii $2(a+5) = 2a+10$
iii $2(a+b) = 2a+2b$
iv $c(a+b) = ac+bc$
v $x(x+y) = x^2+xy$

b What is the quickest way of simplifying:

i $5{\cdot}7 \times 12{\cdot}8 + 7{\cdot}2 \times 5{\cdot}7$ and ii $28{\cdot}96 \times 7 + 28{\cdot}96 \times 17$?

Exercise 6 **a**



Using the dimensions in the drawing, write down the values of the following:

Area A =
Area (A + B) =
Area B =
Area (A + B) − area B =

The final expression contains brackets. Remove these and simplify. Complete the following statement:

$(x+y)(x-y) =$

b Work out the following expressions in two ways, one by using the rule you have just established and the other by multiplying out in the usual way.

i 13^2-12^2 ii 25^2-24^2 iii $7{\cdot}3^2-2{\cdot}7^2$

Exercise 7 **a** Simplify ${}^{-}2 \times {}^{+}3$; $5 \times (-4)$; $a \times (-b)$; $x \times {}^{-}y$.

b Multiply up $(a+b)(c+d)$, $(4+x)(3+y)$.

c Remove the brackets from the following by using the rule mentioned at the beginning of this section.

i $(4+x)(3-y)$
ii $(7-a)(4+b)$
iii $(x+y)(x-y)$
iv $(a-b)(a+b)$

d Put the following into brackets, ( )( ):

i $a(x+y)-b(x+y)$
ii $x(x+y)-y(x+y)$
iii x^2-y^2
iv a^2-b^2

e When an expression is replaced by the product of two expressions, it is said to be 'factored' (or 'factorized'). For instance, if $ac+bd+ad+bc$ is rewritten $(a+b)(c+d)$, it has been factored. The quantities $(a+b)$ and $(c+d)$ are the factors of the expression. This is like replacing 51 as 3×17, which is also factoring. Factor the following expressions:

i 17^2-15^2
ii $17{\cdot}6^2-16{\cdot}6^2$
iii $456{\cdot}3^2-454{\cdot}3^2$
iv $789{\cdot}5^2-789{\cdot}3^2$

Use your factors to simplify the expressions.

f Rewrite each of the following without the brackets.

i $(2x+y)(2x-y)$
ii $(3x+2y)(3x-2y)$
iii $(7x+5y)(7x-5y)$
iv $(15a-7b)(15a+7b)$
v $(14-9p)(14+9p)$
vi $(16+61q)(16-61q)$
vii $(\sqrt{2}+4)(\sqrt{2}-4)$
viii $(\sqrt{17}+\sqrt{5})(\sqrt{17}-\sqrt{5})$
ix $(v^2-w^3)(v^2+w^3)$
x $(x^0+y^0)(x^0-y^0)$

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g In your answers to the last set of questions (f) parts (vii) and (viii) simplify further. Simplify them.

h The question (f) (x) is rather special. What does it simplify to?

i Are $(a+b)$ and $(a-b)$ in the above work numbers? What kind of numbers? If we say that $(a+b)(a-b) = (a-b)(a+b)$, what principle are we using?

Exercise 8 **a** i What expression is equal to $a(c+d)$?

ii If $a = 7$, $c = 20$, and $d = 3$, what calculation does (i) resemble? Explain.

b i Complete the sentence $(a+b)^2 =$

ii If $a = 100$ and $b = 0{\cdot}0001$, use the RHS of your expression to work it out.

c Complete this sentence: $x^2+2xy+y^2 =$

d Write down the solution set of the following:

$\{(x, y) | x^2+2xy+y^2 = 9; x, y \text{ are natural numbers}\}$

e Two numbers multiplied to give a number are factors of the number. Thus 7 and 3 are factors of 21. What are the factors of

i $(c+d)^2$
ii e^2+f^2+2ef?

f What are the factors of:

i a^2
ii ab
iii $x(y+z)$
iv $pm+pn$?

g Numbers can also have three factors. 7, 3, 2 are thus factors of 42. Since they are prime numbers, they are prime factors. What are the prime factors of

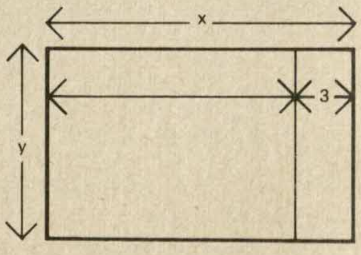
i 70
ii 42?

h What are the factors of

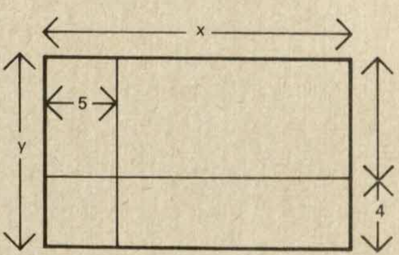
i $2ab$ ii $2b^2$ iii b^3?

Exercise 9 **a**

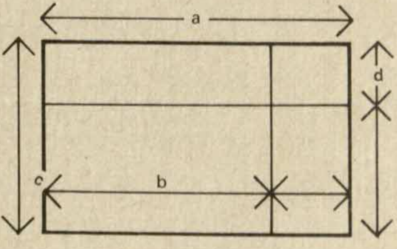
A



B



C



In each of the three diagrams, some measurements of the rectangles are not given. On your own copy of them fill in the missing measurements.

b In diagram A write down the area of the whole rectangle and also the sum of its parts. Are these expressions equal?

c Repeat this for diagrams B and C.

d In the last expression put $a = c$ and $b = d$. What new expression do you get?

Answers
Exercise 9d

$$c^2 = d^2 + 2d(c-d) + (c-d)^2 \Leftrightarrow (c-d)^2 = c^2 + d^2 - 2cd$$

$$\text{or } c^2 = d^2 + (c-d)(2d + c - d)$$

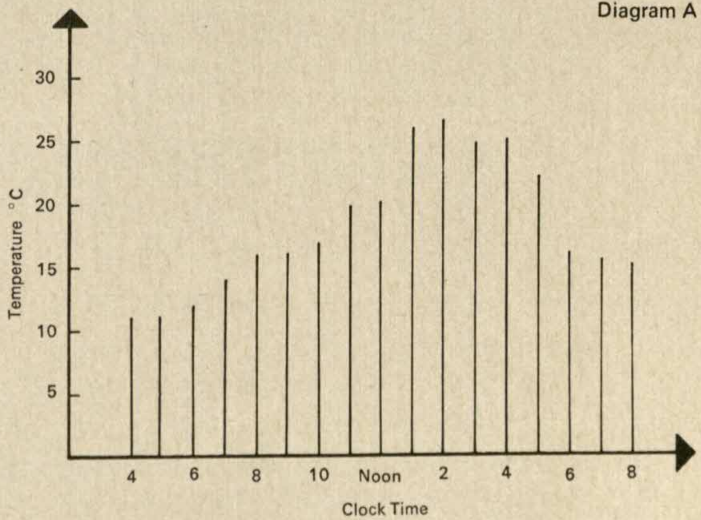
$$\Leftrightarrow c^2 - d^2 = (c-d)(c+d)$$

More work on statistics

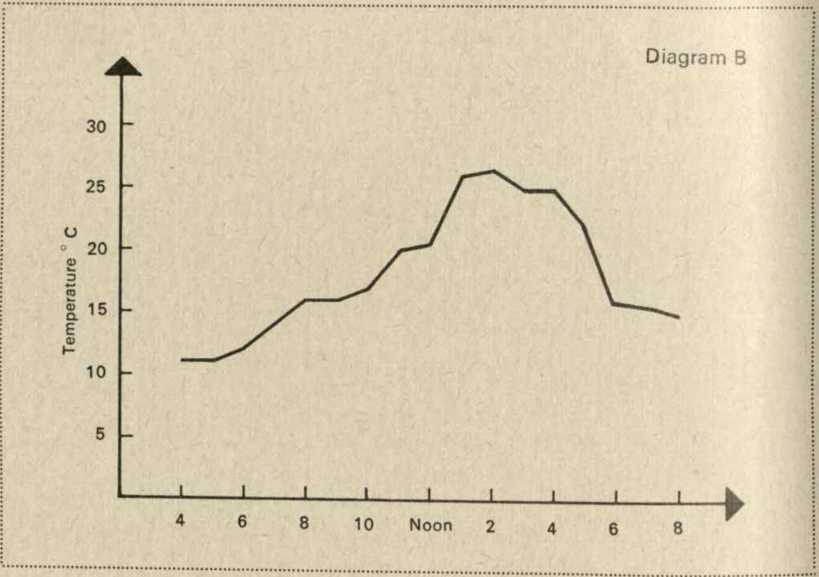
We have already met pie-charts and column-graphs. Now we shall look at several other points about graphs of statistical quantities. The temperature on a breezy day in July was measured and the following results were obtained by taking the temperature each hour from 4 a.m. to 8 p.m.

Time	Temperature °C.	Time	Temperature °C.
4 a.m.	11	noon	21
5	11	1 p.m.	26
6	12	2	27·5
7	14	3	25
8	16	4	25
9	16	5	22
10	17	6	16
11	20	7	15·5
		8	15

If we draw a column-graph of these figures, we obtain a graph like this.



The vertical lines do not do anything in the graph except indicate that a reading is being represented and to indicate the position of the top. This is rather like the idea of a schoolboy as an object for keeping his school-cap off the floor! Why not represent the graph by the positions of the top point alone? If we do, we can make the diagram clear by joining the points with straight lines. Do you think that the intermediate positions along the lines are accurate? (See the diagram B.)



Exercise 1 **a** Draw some temperature graphs for yourself. Measure the temperature at hourly intervals in (i) your classroom, (ii) the school yard, (iii) your living room, and use these results.

b Choose some convenient point in your school and count the numbers of people who pass by in quarter-hour intervals. You cannot sit there all day so choose certain times to do it, some of them when everyone is supposed to be working.

Exercise 2 **a** The lines in these graphs 'zig-zag' about. Why do you think this is?

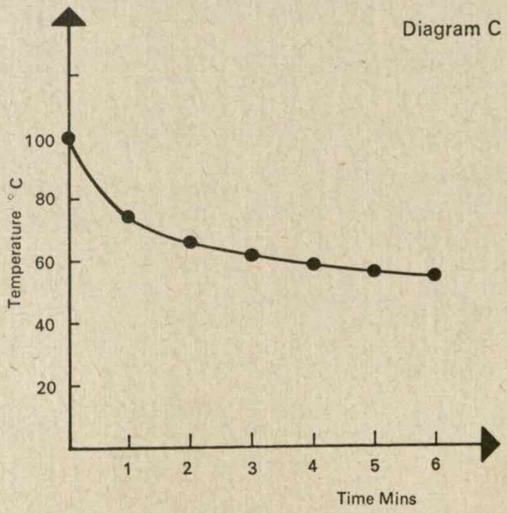
b Is there a temperature at all times? Could you draw a temperature graph which shows the temperature accurately at all times? Can it be done by any means at all?

c Is it possible with certain things which vary, to make measurements only at certain times but to be fairly sure of what the intermediate figures are? When do you think you might be able to do this?

Here is a set of temperatures again. This time the temperature is of a metal can containing hot water. The can is protected from draughts and is allowed to cool, the temperature being taken at one minute intervals.

Time elapsed (minutes)	Temperature °C.
0	100
1	75
2	66
3	62
4	58
5	56
6	55

These points have been plotted on diagram C.



Exercise 3 Measure the temperatures at various distances (about 1 ft. apart) from a fire. (An electric fire will do very well.) Draw a graph of these distances and temperatures.

Exercise 4 a In these graphs we have drawn smooth curves through the points we have plotted. Why do you think we can do this? Is it possible that we are wrong to do so?

b In diagram C why does the curve drop very steeply to begin with and later become almost level? Can you give physical reasons why this should be so?

c Carry out an experiment on the cooling of water – a kettle full of nearly boiling water will do, but be very careful – taking the temperature every half minute. Plot the values you obtain and join with a smooth curve.

Exercise 5 A group of campers established a camp from Monday to Friday. They had a communal kitchen, and unknown to them the organizer counted them in to certain meals. He took the average numbers over certain days and drew block graphs. In diagram D we show his breakfast-graph. In diagram E we have his average numbers for supper time (not including Thursday). His figures for Thursday are represented in diagram F.

Look carefully at the diagrams and answer the questions which follow.

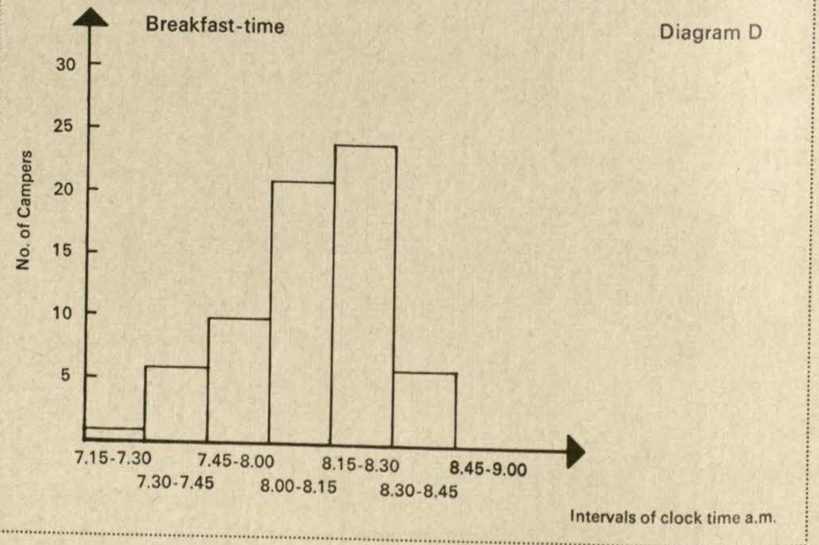
a We do not know how many campers there were. Why not?

b What is the least number there must have been?

c Assuming breakfast began at 7.30 and lasted until 8.30, would you say that the campers were keen on early rising?

d How many tried to get breakfast late?

e On most days the campers went out hiking all day, taking a sandwich lunch with them. On Thursday they went out as usual but one party



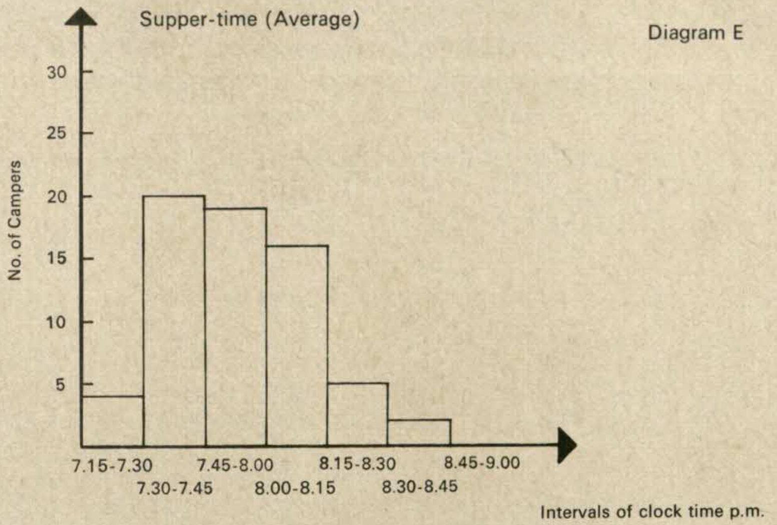


Diagram E

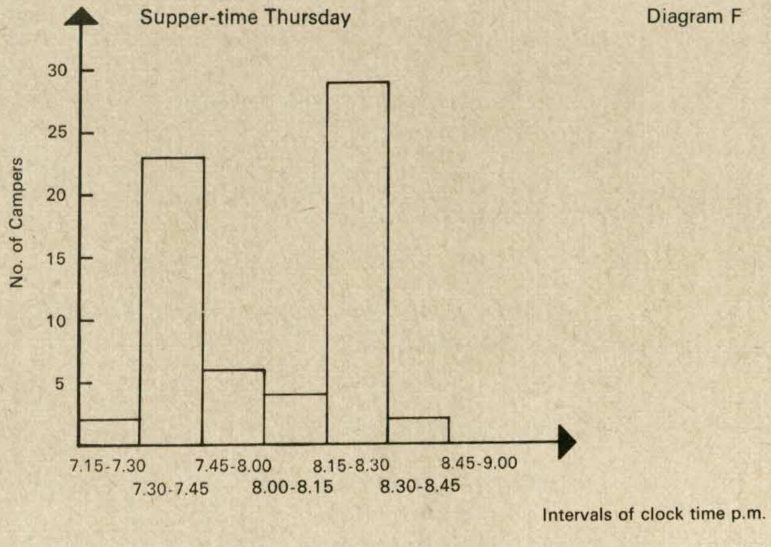


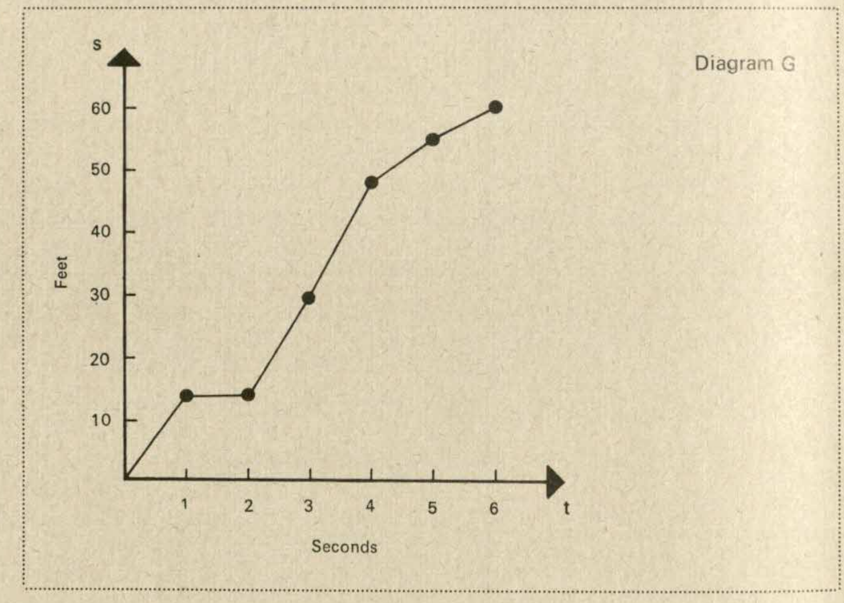
Diagram F

decided on an extra long walk. How many do you think were probably in this party?

f Would you say that the graphs gave any indication about whether or not the hikers had a long walk on the other days?

Exercise 6 Go to a local cross roads, traffic island or main road and count the numbers of vehicles which pass during quarter hour intervals at various times of day. Keep separate records of private cars and commercial vehicles. Draw block graphs to indicate the results and decide what conclusions you can come to.

Exercise 7 **a**



In diagram G, we show a graph of the distances moved by a car over a period of 6 seconds.

How many feet did the car cover in each successive second?

b What is the significance of the line parallel to the time-axis between seconds 1 and 2?

c During which period was the vehicle moving fastest?

d Do you think that in general the vehicle was moving very fast?

e Does the graph give you any certain information about the kind of journey covered?

f About how many feet had the vehicle covered after $4\frac{1}{2}$ seconds?

Exercise 8 The following figures indicate distances covered by a car in a period of 6 seconds. Use them to draw a graph. Do you think you should join the points as in diagram G or with a smooth line?

Time (secs)	Distance from start (ft.)
0	0
1	10
2	14
3	30
4	32
5	50
6	54

a During which second did the car travel fastest?

b Was the vehicle stationary for a whole second? (Be careful how you answer this!)

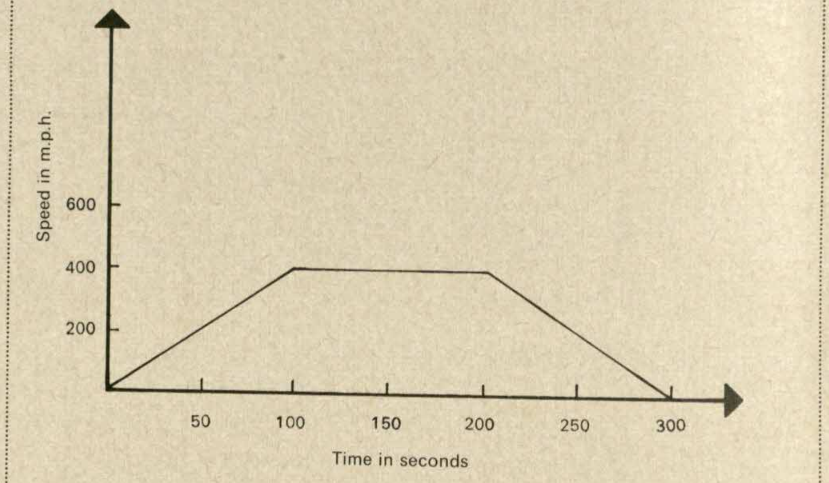
c Was it moving very slowly for a whole second at any time?

d How far did the car travel in the 6 seconds?

e Suppose it had travelled at a steady rate of 9 ft. every second, how far would it have gone in the time?

Speeds

When you ride a bicycle, do you travel at a steady rate all the time? Of course not. For one thing when you set off, you begin from rest and increase your rate of moving up to your normal speed. As you go along, you find that you have to change speed to go round bends, to stop for obstructions or traffic lights or to go through narrow gateways. Yet you certainly have a rate of motion which you call your speed. When Campbell made his record run recently in Australia, he got up speed, then he was timed between two fixed markers. This does not mean that he was moving at a steady rate the whole time between the markers: it varied a little. But we will suppose that he did maintain a steady speed. If so we could draw a graph of his speed against time. It might look like this:



If his speed was not quite steady, the horizontal line would be not quite straight.

When we measure speed, we do two things. We assume that for some interval of time, however short, the speed is steady and we then say what distance has been covered in that time. If a car travels a distance of 22 ft. in $\frac{1}{2}$ second, the speed could be given as 22 ft. per half-second. Another car might travel 33 ft. in $\frac{3}{4}$ second. How can we compare these? It would be better to scale them up to a whole second, wouldn't it?

The first speed then becomes 44 ft. per second. The other is 11 ft. every $\frac{1}{4}$ second or 44 ft. per second. We can compare them now and find them to be equal. We do not usually give the speeds of motor vehicles in ft. per second; more often we use miles per hour. How can we convert them?

44 ft. per second is 44×60 ft. per minute or $44 \times 60 \times 60$ ft. per hour. (Where do the two sixties come from?) This is the same as $\frac{44 \times 60 \times 60}{5280}$ miles per hour. (Where does the 5280 come from?)

Exercise 1 **a** Work out 44 ft. per second as miles per hour.

b From the beginning work out 88 ft. per second as miles per hour. The answer should of course come to twice the previous value.

c Calculate each of the following speeds as miles per hour by the same method.

i 60 ft. per second
ii 100 ft. per second
iii 1000 ft. per second
iv 1320 ft. per second
v 55 ft. per second.

d Make a list of the speeds in miles per hour equivalent to:

ft. per second	ft. per second
88	22
44	11

Can you see how to use this table to convert 55 ft. per second to miles per hour?

e For this question you may wish to extend your table. Convert each of the following to the units in brackets.

i $1\frac{7}{8}$ m.p.h. (f.p.s.)
ii 352 f.p.s. (m.p.h.)
iii $181\frac{1}{2}$ f.p.s. (m.p.h.).
iv $11\frac{1}{4}$ m.p.h. (f.p.s.)
v $108\frac{3}{4}$ m.p.h. (f.p.s.)

f Use the method at the beginning of this exercise to convert 100 cm. per second to kilometres per hour.

g A boy on a bicycle covers 352 yds. in a minute. How fast is he going in miles per hour? Is this a reasonable speed for a bicycle?

Exercise 2 **a** Look back to diagram G in the previous section. How far did the vehicle travel in 6 seconds? What speed is this in ft. per second? What speed in miles per hour? If the vehicle had been able to travel at nearly 7 m.p.h. steadily would it have covered about the same distance in 6 seconds? How fast exactly would it have to have travelled to cover the same distance precisely? (This speed is called the *average speed* for the journey.)

b Look back at the graph you drew for Exercise 8 of the previous section. Find the average speed in this case.

As you will see, to work out average speed you must take the total distance covered and the total time taken.

Exercise 3 **a** Look at diagram G again. Work out the average speed for the following parts of the journey.

i 0–1 secs.	v 0–5 secs.	viii 1–4 secs.
ii 0–2 secs.	vi 0–6 secs.	ix 3–6 secs.
iii 0–3 secs.	vii 2–5 secs.	x 4–6 secs.
iv 0–4 secs.		

b Using the values of Exercise 8 in the last section, find the average speeds for the same parts of this journey.

Exercise 4 Diagrams H, I, K, and L show graphs of speed against time.

a Explain in words the kind of motion which each illustrates.

b In diagram I, on the solid line, what is (i) the speed at 3 seconds and (ii) the time when the speed is 40 ft. per second?

c In diagram J what is (i) the time when the speed is 20 ft. per second and (ii) the speed at 5 seconds?

d In diagram K (i) at what two times is the moving vehicle stationary and (ii) what is its maximum speed?

We have been referring to 'speed' which is the common word which you use to describe rate of moving. Mathematicians find it useful to distinguish two separate kinds of speed. One is speed in a straight line which is called *velocity*; the other is speed not in a straight line which is referred to as *speed*. Any change in velocity is called *acceleration*; acceleration can thus be a change of direction as easily as a change of speed. This will be important later.

Exercise 5 **a** Look again at diagram I. What can you say about the accelerations represented by the solid line and the dashed line? (Assume that the motion is in a straight line.)

b What is the acceleration of this vehicle if the motion is in a straight line?

c What is happening to the acceleration in diagram J?

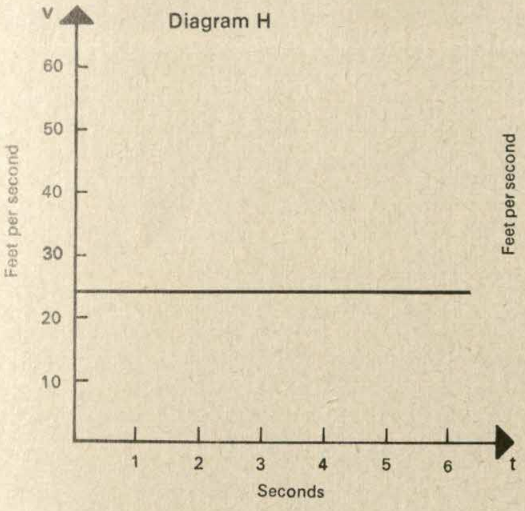


Diagram H

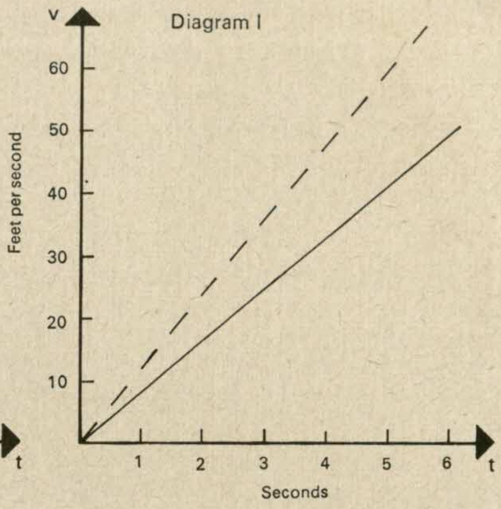


Diagram I

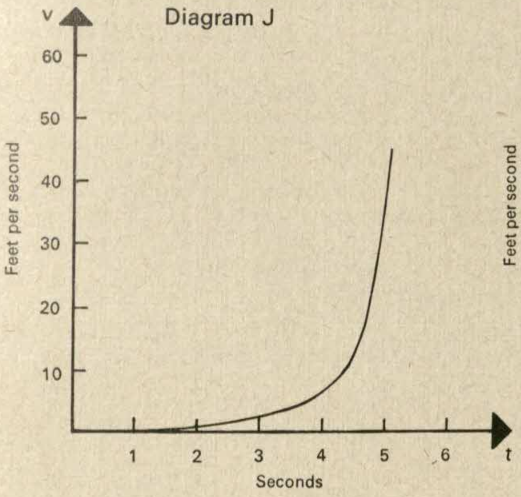


Diagram J

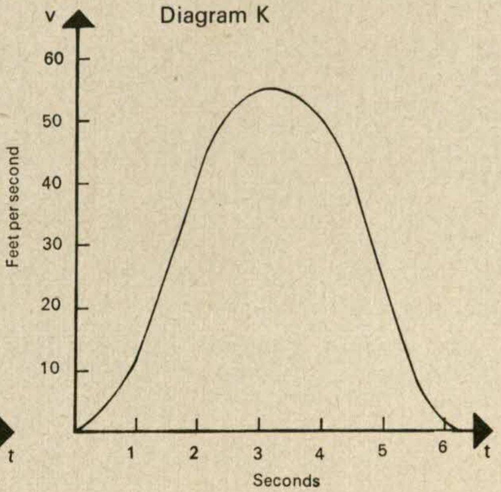


Diagram K

d Describe carefully the acceleration in diagram K. (The motion represented is in a straight line.)

Exercise 6 Refer to diagrams L and M. We shall use the abbreviations $s(t)$ and $v(t)$ to refer to distance–time and velocity–time graphs respectively. The heavy lines in each of these graphs represent the same motion.

Diagram L

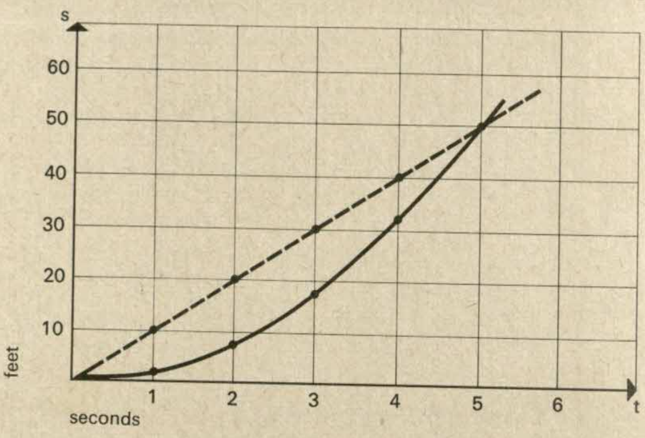
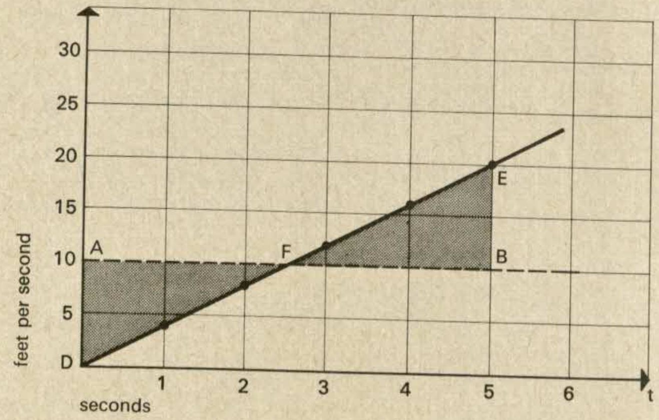


Diagram M



a On the $s(t)$ graph, the dashed line and the solid line represent two motions. What can you say about (i) the velocities in the two cases, and (ii) the distances covered in 5 seconds by each?

b On the $s(t)$ graph, what is the average speed of the two motions during the first 5 seconds?

c On the $v(t)$ graph, what can you say about the acceleration represented by the solid line?

When you want to calculate the velocity of an object, you consider the distance it has travelled in a given time. For example if an object covers 20 ft. in 5 seconds you know that it is travelling at a velocity of 4 ft. per second – *if the velocity is steady*. If it is not a steady motion, then what you have calculated is the average velocity, or that steady velocity which would cause the object to cover the same distance in 20 seconds. You have noticed that we have used the letter s to stand for distance, v for velocity and t for time. We can therefore write: $v = \frac{s}{t}$, giving appropriate units, e.g., f.p.s., m.p.h., etc.

Exercise 7 Look at the table below and complete the empty column:

	s	t	v
a	100 ft.	20 secs.	f.p.s.
b	60 yds.	30 secs.	yds.p.s.
c	51 miles	17 min.	m.p.h.
d	513 miles	30 hrs.	m.p.h.
e	$2\frac{7}{8}$ ft.	$\frac{3}{4}$ sec	f.p.s.

If you look at your answers to (a), (b), (d), and (e), you will be able to see relations between the numbers in the columns. Is it also true that $s = vt$ and $t = \frac{s}{v}$? Check these formulae from the values in the table. Why did we not consider (c)? How can we bring this into line with others in this respect?

Exercise 8 **a** A motorist was to make a journey from Shrewsbury to Worcester and he planned to average 25 m.p.h. over the whole journey. His map gave him the following information:

Shrewsbury
12 miles
Much Wenlock
8 miles
Bridgnorth
13 miles
Kidderminster
$14\frac{3}{4}$ miles
Worcester

Diagram L

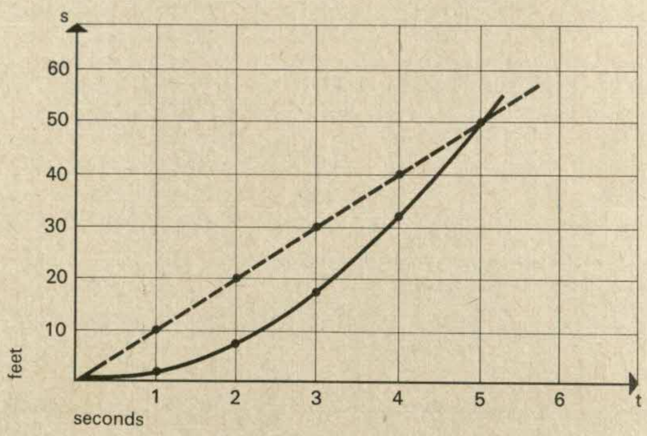
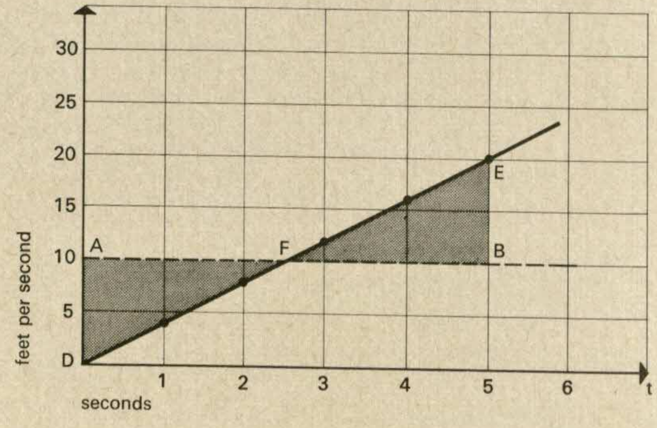


Diagram M



a On the $s(t)$ graph, the dashed line and the solid line represent two motions. What can you say about (i) the velocities in the two cases, and (ii) the distances covered in 5 seconds by each?

b On the $s(t)$ graph, what is the average speed of the two motions during the first 5 seconds?

c On the $v(t)$ graph, what can you say about the acceleration represented by the solid line?

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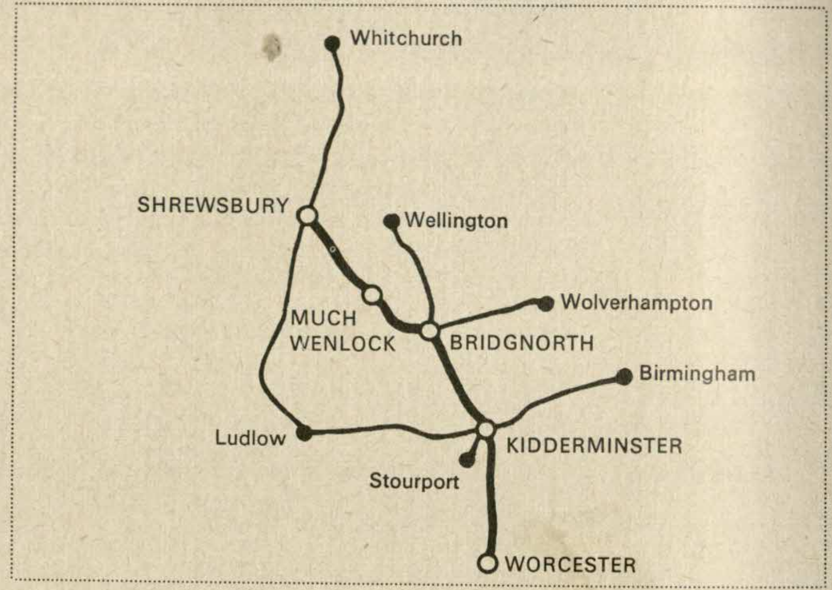
Exercise 8 **a** A motorist was to make a journey from Shrewsbury to Worcester and he planned to average 25 m.p.h. over the whole journey. His map gave him the following information:

Shrewsbury	12 miles
Much Wenlock	8 miles
Bridgnorth	13 miles
Kidderminster	$14\frac{3}{4}$ miles
Worcester	



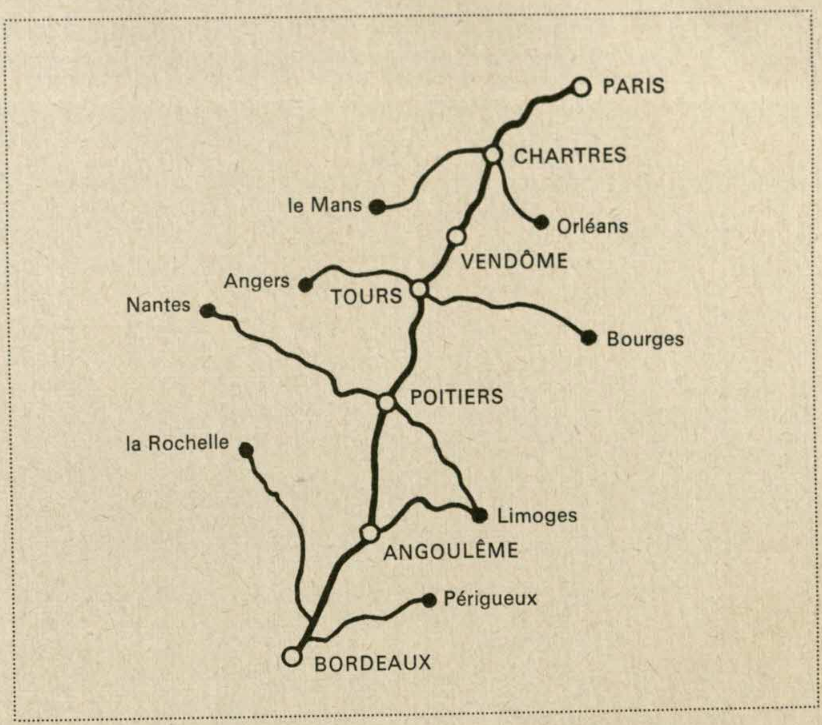
Find out how long he would have had to take over the journey (to one decimal place of a minute) to maintain the speed he wished to. Do you think a calculation like this should be done to one decimal place of a minute? or to the nearest minute?

b When the motorist actually did his journey, he found that he did the first part of the journey at a speed above his intended average. He got from Shrewsbury to Bridgnorth in 40 minutes. However, between Bridgnorth and Kidderminster he had a breakdown which delayed him by exactly half an hour. He broke down $7\frac{3}{4}$ miles after he had passed through Bridgnorth, and had taken $23\frac{1}{4}$ minutes from there. How fast would he have to travel over the rest of the journey to make the average he intended? (Give the answer to the nearest m.p.h.) Could he do it?

c From a suitable map find out how long it would take at an average speed of 20 m.p.h. from Shrewsbury to Aberystwyth by the most direct main-road route.

d I am planning a continental route part of which is along International Route 3 from Paris to Bordeaux. The distances are as follows:

Paris
94
Chartres
83
Vendôme
102
Poitiers
109
Angoulême
116
Bordeaux



These distances are of course in kilometres. Taking 50 km. as equivalent to 30 miles, find out how long it will take if I estimate that I can maintain an average speed of 30 m.p.h.

Turn again to diagrams L and M. As we have already said the solid lines represent the same motion. From the first we have already seen that the average velocity is 10 ft. per second. If we plot this onto diagram M we obtain the dashed 'horizontal' line. This average velocity accomplished the same end result as the solid line on this graph.

Exercise 9 **a** What do you notice about the areas of $\triangle AFD$ and $\triangle FEB$?

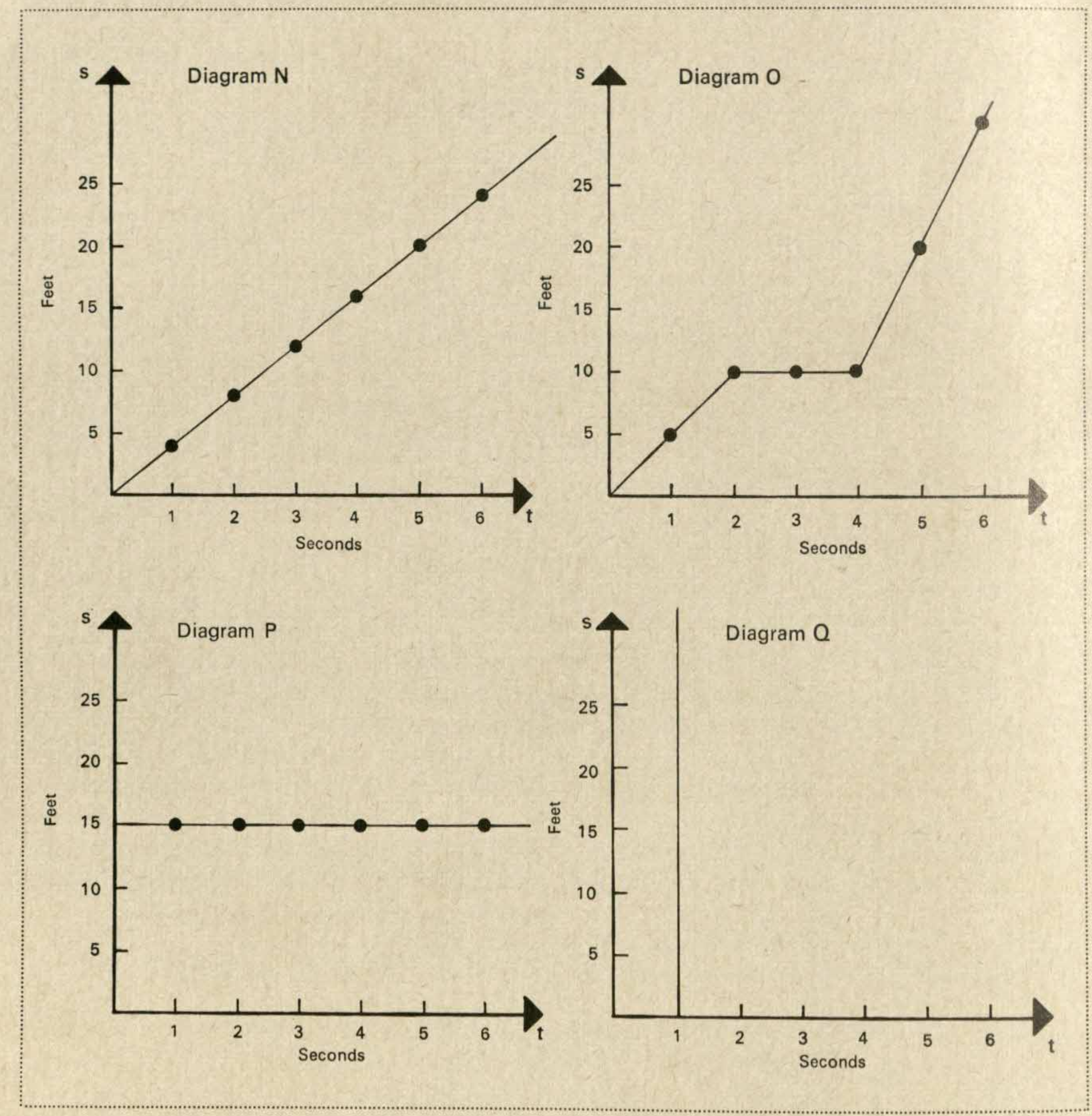
b What do you notice about the areas ABCD and $\triangle DEC$? Can you see two ways of discovering this fact?

c What is the area of the rectangle ABCD?

d From diagram L, what is the distance covered in this motion?

You have just discovered a connection which you will be meeting again later on. The measure of the area under a $v(t)$ graph gives a measure of the distance covered by the motion.

Exercise 10 **a** Describe in words the motion represented on diagram N. What can you say about the average speed at any stage in the motion?



b Describe the motion represented in diagram O, explaining the significance of the three sections.

c What sort of motion is represented in diagram P?

d What do you think diagram Q means? Can it actually happen?

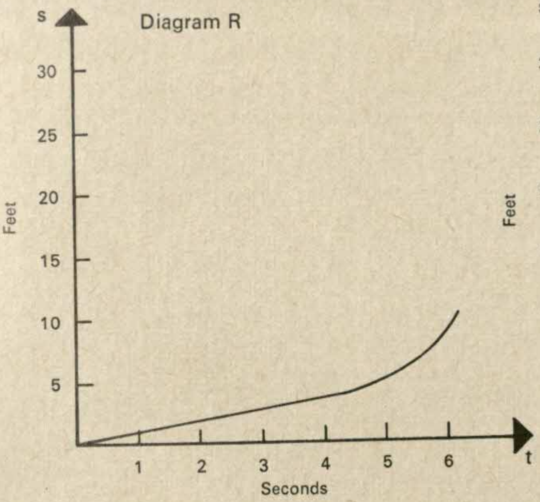


Diagram R

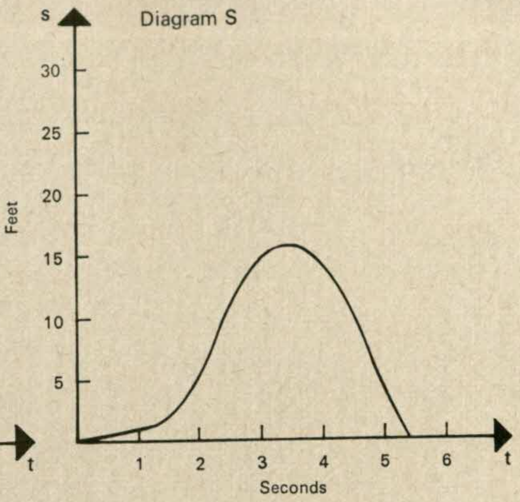


Diagram S

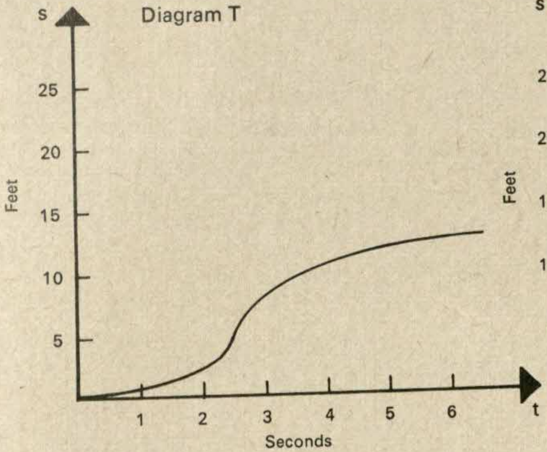


Diagram T

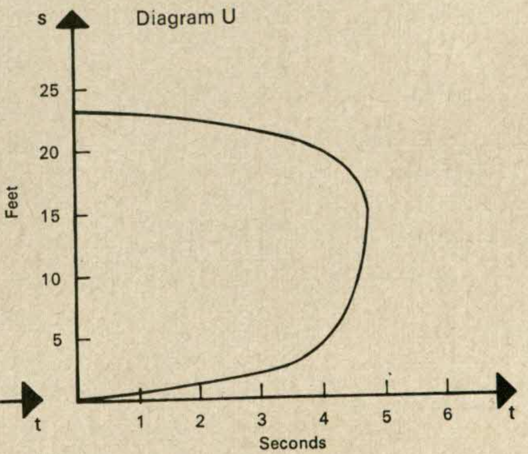


Diagram U

e Describe the motion in diagram R. In practice, could this kind of motion continue as it has been begun?

f Describe the motion represented in diagram S. What is the greatest distance the object reaches? What happens after that? When is the object at its starting point?

g Describe the motion in diagram T.

h What comment can you make about the graph in diagram U?

Section 28 **Numbers, powers and square roots again**

Exercise 1 Find natural numbers to put into the frames in the following statements. (Remember that if a frame appears twice in the same statement, it must contain the same number each time.)

a $\sqrt{169} = \triangledown$

b $\sqrt{\square} = 17$

c $\sqrt{16} + \sqrt{\triangle} = \sqrt{25}$

d $\square^5 \times \square^4 = 6^{\triangledown}$

e $\square^2 \times \square^3 = \square^5$

Exercise 2 **a** Simplify the following by writing with a single power.

i $4^5 \times 4^3$
ii $6^3 \times 6^3$
iii $(4^2)^2$
iv $5^1 \times 5^7$
v $10^4 \times 10^2$
vi $10^7 \times 10^4$
vii $8^4 \times 8^3 \times 8^7$
viii $a^3 \times a^3$
xi $b^4 \times b^4$
x $(c^7)^2$

b Find integers to put into the frames in the following.

i $4^{\square} \times 4^5 = 4^{10}$
ii $a^6 \times a^{\triangledown} = a^{12}$
iii $(b^7)^{\square} = b^{14}$
iv $(c^{\triangledown})^2 = c^{18}$
v $(d^{\triangledown})^2 = d^5 \times d^7$
vi $e^5 \times e^9 = (e^{\triangle})^2$
vii $f^4 \times f^4 = (f^{\square})^2$
viii $g^{2a} = (g^{\triangledown})^2$

c Write the following results in power or index notation.

i $\sqrt{10^2}$
ii $\sqrt{10^4}$
iii $\sqrt{2^4}$
iv $\sqrt{2^6}$
v $\sqrt{3^8}$
vi $\sqrt{a^{10}}$
vii $\sqrt{a^{2b}}$
viii $\sqrt{a^b}$

d Find numbers to fill the frames in the following.

i $\sqrt{10^{\square}} = 10$
ii $\sqrt{4^{\triangledown}} = 4^5$
iii $\sqrt{a^{\triangle}} = a^6$
iv $\sqrt{b^{\square}} = b^4 \times b^1$

Exercise 3 **a** If k is any integer, is $2k$ even or odd?

b If c is any integer, can you say whether $2c+1$ is even or odd?

c Can you always find the square root of a number a^{2k}?

d How do you think you can deal with a number b^{2c+1} to find its square root?

e Is there any integer which is the square root of (i) 2^3 or (ii) 4^5?

As you have seen it is a simple matter to write down the square root of any number which can be written as an even power of something. For instance $4 = 2^2$ so $\sqrt{4} = 2^1$ or 2; $16 = 2^4$ so $\sqrt{16} = 2^2$ or 4; $81 = 9^2$ of 3^4 so $\sqrt{81} = 9$ or 3^2. Now $1000 = 10^3$ so that its square root is more difficult. In index notation we may agree to write $\sqrt{10^3} = 10^{\frac{3}{2}}$ but we do not know what this means in 'ordinary' numbers.

If a^2 is the square root of a^4 it follows that $a^2 \times a^2 = a^4$, which we already know is true. Would it be equally reasonable to write $10^{\frac{3}{2}} \times 10^{\frac{3}{2}} = 10^3$? Does this fit the rule for indices? Does $10^{\frac{5}{2}} \times 10^{\frac{5}{2}}$ also fit the rule for indices? We shall have to check the rules very carefully to ensure that this notation makes sense. If it does we shall use it; if not we shall have to reject it and seek a better way.

Exercise 4 Throughout the following exercise fill in the frames according to the usual rules.

a i $\sqrt{16} = \sqrt{2^4} = 2^{\square} = \triangle$

ii $\sqrt{4} \times \sqrt{4} = \square \times \square = 2^{\triangle} = \nabla$

iii $\sqrt{4} \times \sqrt{4} = \sqrt{\square}$

b i $\sqrt{256} = \sqrt{2^{\nabla}} = 2^{\square} = \bigcirc$

ii $\sqrt{16} \times \sqrt{16} = \nabla \times \nabla = 2^{\square} = \triangle$

iii $\sqrt{16} \times \sqrt{16} = \sqrt{\boxed{}}$

c i $\sqrt{81} = \sqrt{3^{\nabla}} = 3^{\square} = \triangle$

ii $\sqrt{9} \times \sqrt{9} = \square \times \square = 3^{\triangle} = \nabla$

iii $\sqrt{9} \times \sqrt{9} = \sqrt{\boxed{}}$

d i $\sqrt{5} \times \sqrt{5} = \sqrt{\boxed{}}$

ii $\sqrt{71} \times \sqrt{71} = \sqrt{\boxed{}}$

iii $\sqrt{a} \times \sqrt{a} = \sqrt{\triangle^2} = \triangle$

iv $\sqrt{10^5} \times \sqrt{10^5} = \sqrt{10^{\square}} = \square$

e i $\sqrt{400} = \sqrt{\nabla^2} = \square$

ii $\sqrt{16} \times \sqrt{25} = \triangle \times \square = \boxed{}$

iii $\sqrt{144} = \sqrt{\nabla^2} = \triangle$

iv $\sqrt{9} \times \sqrt{16} = \triangle \times \square = \boxed{}$

v $\sqrt{4} \times \sqrt{36} = \nabla \times \square = \boxed{}$

vi $\sqrt{144} = \sqrt{9} \times \sqrt{\nabla} = \sqrt{4} \times \sqrt{\boxed{}}$

vii $\sqrt{16} \times \sqrt{\triangle} = \sqrt{64}$

f i $\sqrt{8} = \sqrt{4 \times \square} = \sqrt{4} \times \sqrt{\nabla} = 2\sqrt{2}$

ii $\sqrt{640} = 8\sqrt{\square}$

iii $\sqrt{50} = \sqrt{\square} \times \sqrt{10}$

iv $\sqrt{1000} = \nabla \times \sqrt{10}$

v $\sqrt{10^3} = \sqrt{10^2 \times 10}$
$= \nabla \times \sqrt{10}$

g i $\sqrt{10^3} \times \sqrt{10^5} = \sqrt{10^{\boxed{}}} = 10^{\triangle}$; $10^{\frac{3}{2}} \times 10^{\frac{5}{2}} = 10^{\square} = 10^{\nabla}$

ii $\sqrt{10} \times \sqrt{10^9} = \sqrt{10^{\square}} = 10^{\boxed{}}$; $10^{\frac{1}{2}} \times 10^{\nabla} = 10^{\frac{10}{2}} = 10^{\triangle}$

iii $\sqrt{2} \times \sqrt{2^3} = \sqrt{2^{\square}} = 2^{\triangle}$; $2^{\frac{1}{2}} \times 2^{\frac{3}{2}} = 2^{\nabla}$

iv $\sqrt{3^7} \times \sqrt{3^3} = 3^{\triangle} = 3^{\nabla}$; $3^{\square} \times 3^{\frac{3}{2}} = 3^5$

v $\sqrt{a^5} \times \sqrt{a^3} = \sqrt{a^{\nabla}} = a^{\triangle}$; $a^{\frac{5}{2}} \times a^{\frac{3}{2}} = a^{\square}$

From this work do you think that $10^{\frac{1}{2}}$ and $10^{\frac{3}{2}}$ in fact $10^{(\text{any fraction})}$ obeys the usual multiplication rule for indices?

Exercise 5 **a** $4^{\frac{1}{2}} \div 4^{\frac{1}{2}} = 4^0 = \dfrac{4^{\frac{1}{2}}}{4^{\frac{1}{2}}} = \boxed{}$

b $16^{\frac{1}{2}} \div 16^{\frac{1}{2}} = 16^{\square} = \triangledown$

c i $\sqrt{\dfrac{16}{4}} = \sqrt{4} = \square$

ii $\sqrt{\dfrac{16}{4}} = \dfrac{\sqrt{16}}{\sqrt{4}} = \dfrac{\square}{\triangle} = \boxed{}$

iii $\sqrt{\dfrac{100}{25}} = \sqrt{\square}$ or $\dfrac{\sqrt{\boxed{}}}{\sqrt{\triangle}} = \dfrac{\triangledown}{\bigcirc} = \text{⏢}$

iv $\sqrt{\dfrac{1}{16}} = \dfrac{1}{\triangle}$ or $\dfrac{\sqrt{1}}{\sqrt{\triangledown}}$

d i $\dfrac{\sqrt{3}}{\sqrt{75}} = \sqrt{\dfrac{3}{75}} = \sqrt{\dfrac{1}{\square}} = \square$

ii $\sqrt{125} \div \sqrt{5} = \boxed{}$

iii $\sqrt{5} \div \sqrt{125} = \square$

e i $10^{\frac{3}{2}} \div 10^{\frac{1}{2}} = \sqrt{10^{\square}} \div \sqrt{10^{\triangledown}} = \sqrt{10^{\square} \div 10^{\triangledown}} = \sqrt{10^{\triangle}} = \boxed{}$

ii $10^{\frac{5}{2}} \div 10^{\frac{1}{2}} = 10^{\square}$

Does the power $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc. also obey the rules for division of numbers in index form?

In fact, these indices obey all the rules for multiplication and division of numbers in index form. We can now use $10^{\frac{3}{2}}$ as another way of writing $\sqrt{1000}$. When we find $\sqrt{1000}$ in 'ordinary' numbers, we find that it is a decimal number which carries on for ever without repeating itself. Such numbers are called *surds* and it can be shown that they are not members of the rational number system. Remember the rational number system consists of those numbers which can be derived from the ratios of integers. Surds cannot be so derived. When Greek mathematicians first discovered these numbers which cannot be written down or even calculated, they were very surprised indeed. But that is another story. Surds are members of the number system which we call the *real number system* and in which there are other numbers which do all the work of the rational numbers.

On p. 100 we have given a table of some of the square roots of numbers. These numbers have all been rounded off to three figures. In the next exercise you will see a method by which we can get the results more accurately than this.

Exercise 6 **a** Write down 1^2 and 2^2.

b Complete the following:

i $\sqrt{1} =$.
ii $\sqrt{4} =$.
iii $\therefore \square < \sqrt{2} < 2$

c Find $(1{\cdot}4)^2$; $(1{\cdot}41)^2$; $(1{\cdot}42)^2$.

d Can you now complete the following so that the value of $\sqrt{2}$ is more accurately seen?

$$\nabla < \sqrt{2} < \square$$

e Find $(1{\cdot}414)^2$ and $(1{\cdot}416)^2$.

f Can you narrow down $\sqrt{2}$ even more now?

Exercise 7 **a** Can you find an approximate value for $\sqrt{1000}$ now by considering:

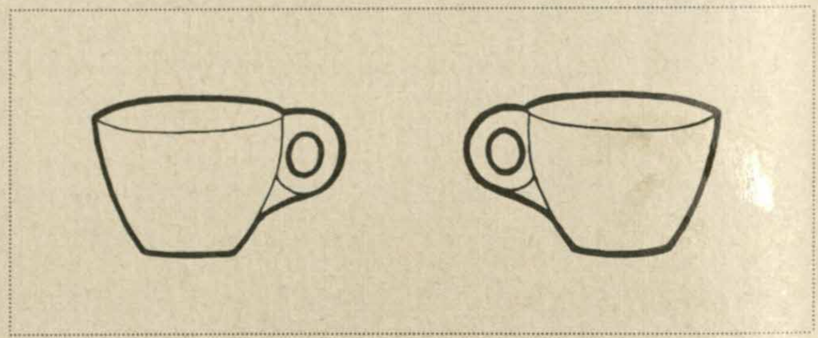
$$(31)^2;\ (32)^2;\ (31{\cdot}6)^2;\ (31{\cdot}7)^2;\ (31{\cdot}62)^2;\ (31{\cdot}63)^2?$$

b Narrow it down further by considering:

$$(31{\cdot}622)^2;\ (31{\cdot}6227)^2;\ (31{\cdot}62278)^2.$$

Section 29 **Rotation**

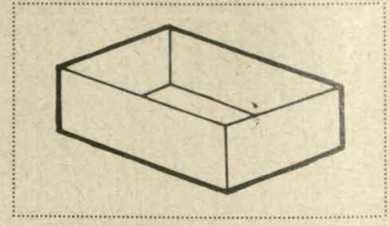
Exercise 1 **a**



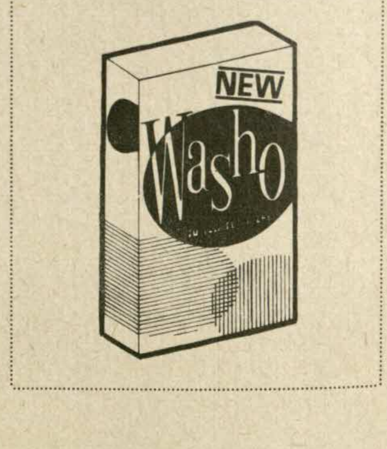
i How could you transform the cup on the right to look like the cup on the left? (Two ways.)

ii Why are left-handed cups suitable for right-handed people?

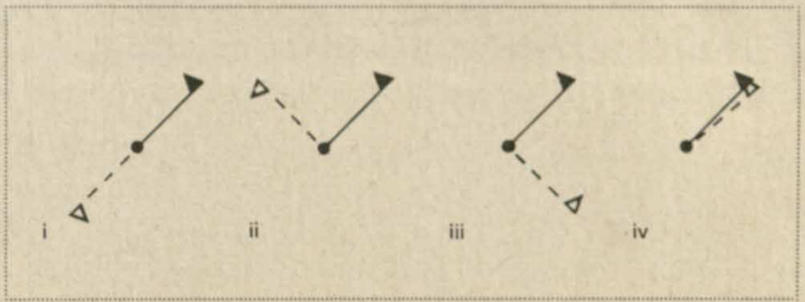
b Can you transform the box to look like itself by (i) reflection and (ii) rotation?



c Can you transform this soap powder box to look like itself by (i) reflection and (ii) rotation? (The design on the back is the same as that on the front.)

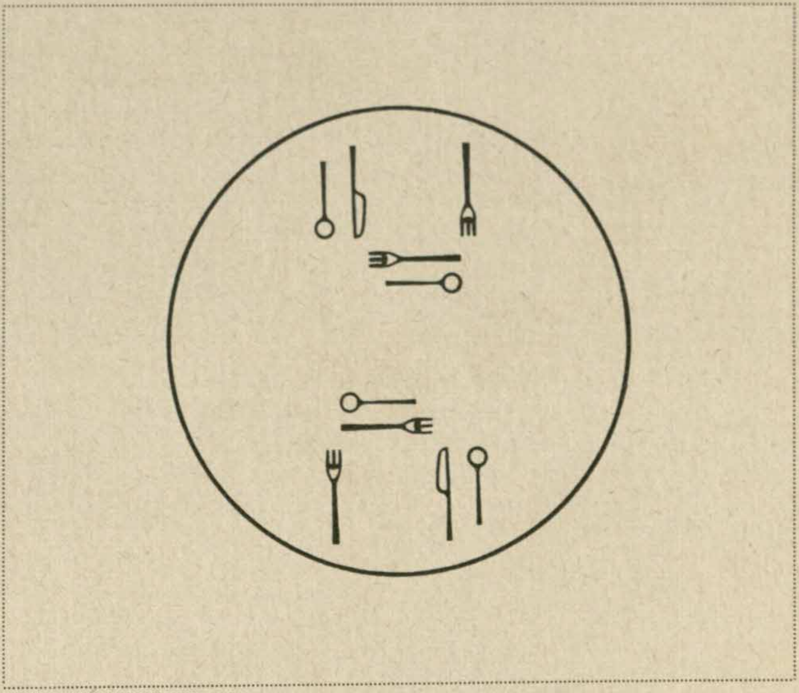


Exercise 2 **a**



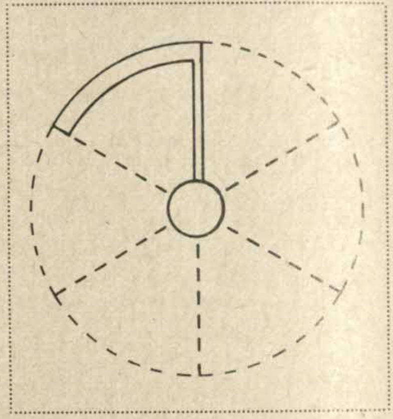
The design can be transformed into the dotted design by rotation about the point. Which of the diagrams show rotations of $\frac{1}{4}$-turn, $\frac{3}{4}$-turn, $\frac{1}{2}$-turn, 1-turn, $1\frac{1}{2}$-turns, $1\frac{3}{4}$-turns, 6-turns, 0-turns? Can any of them be obtained by reflection?

b

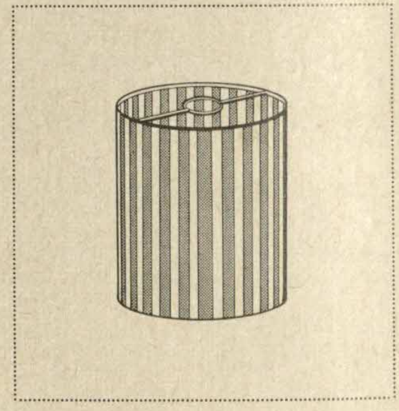


The diagram shows the top of a table laid for lunch for two people. What sort of transformation carries one side into the other? What do you notice about the dessert spoon and fork? Can this transformation be replaced by a reflection?

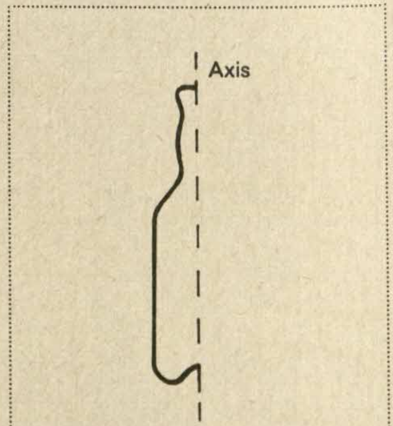
c What rotations will complete the cartwheel? (In degrees.)



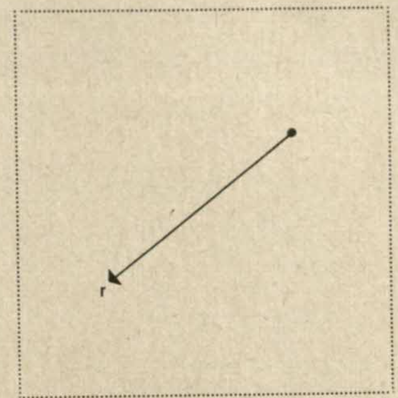
d This cylindrical lampshade is made of 36 strips, alternately red and white. Describe carefully the rotation on which this pattern is based.



e What shape is traced out in space by the rotation of a wire this shape? (We are thinking here of a 3-dimensional locus.)

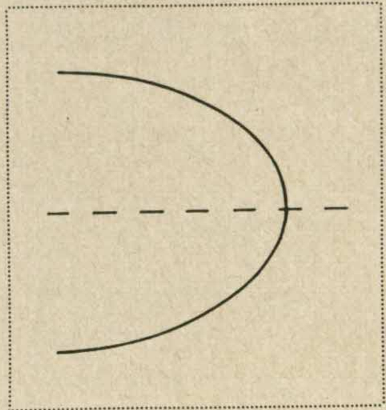


f What geometrical shape is traced out by a complete rotation of half-line *r*? (A 2-dimensional locus.)

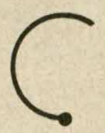


g What geometrical shape is formed by the $\frac{1}{2}$-turn rotation of an isosceles triangle about its axis of symmetry? (A 2-dimensional locus.)

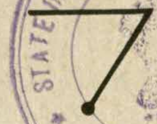
h Where do you find a shape similar to the one formed by the rotation of a parabola like this one about the dotted line as axis? (A 3-dimensional locus.)



Exercise 3 **a**



i



ii

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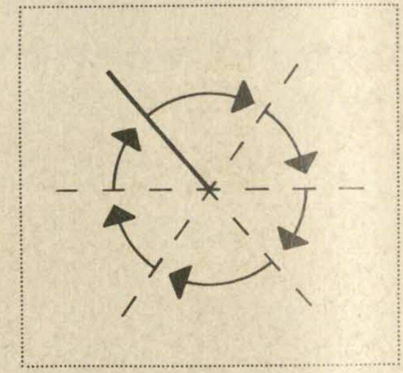
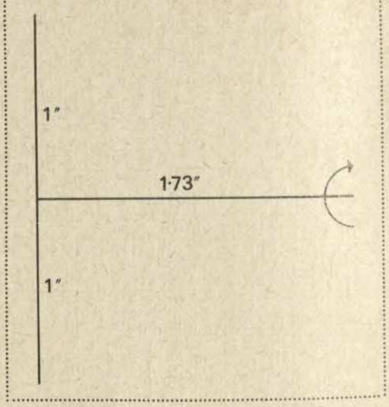
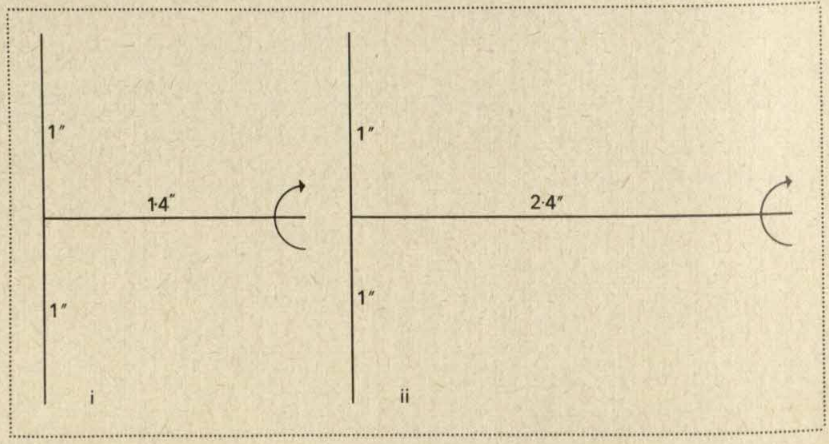
What letters are formed by these figures and their images when each is given a $\frac{1}{2}$-turn about the centre which is marked?

b Through what angle do these rotations (all equal) turn?



c Make careful drawings of the figure formed by the successive rotation through 60° of this figure.

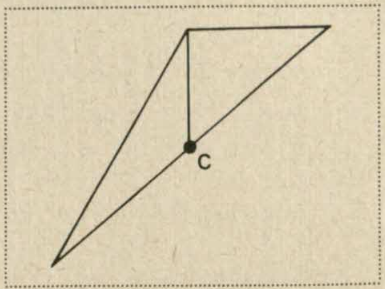




d What figures result from rotations of the same shape (sizes as above) through (i) successive $\frac{1}{5}$ turns, (ii) successive $\frac{1}{8}$ turns?

e As we increase the number of positions in one rotation the figure we obtain gets nearer and nearer to a   . (Complete.)

Exercise 4 Here is a triangle with one vertex joined to the mid-point of the opposite side. (This line is called a *median* of the triangle.)



a Draw a diagram to show what happens when it is rotated half-turn about C.

b On your finished diagram mark in any equal angles.

c Can you see now that there are some parallel lines? Which are they?

d Mark any equal line segments.

e What can you say about the diagonals of your quadrilateral?

f Which triangles in your figure are congruent?

g This figure is called a parallelogram. Why do you think this is its name?

Exercise 5 **a** Can you obtain the parallelogram by reflecting a triangle?

b Can you obtain (i) a rhombus and (ii) a square by rotation?

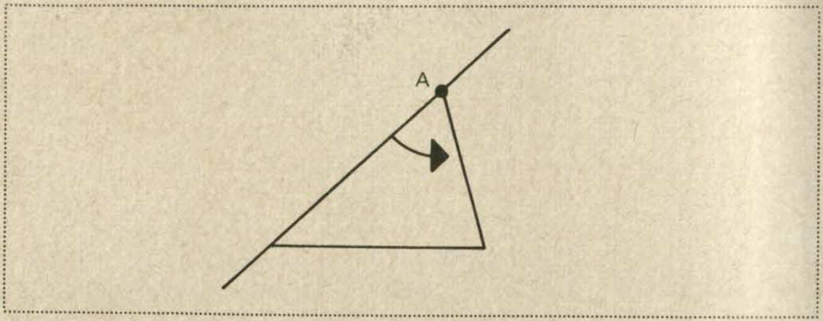
c Would you say that the rhombus and square are special sorts of parallelogram? If so what is special about them?

d Can you identify the following figures?

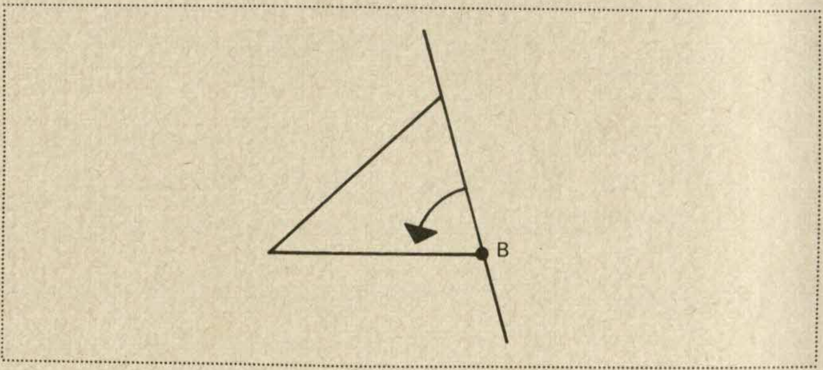
i A quadrilateral with two pairs of opposite sides parallel.
ii A parallelogram with two adjacent sides equal.
iii A parallelogram with two adjacent sides equal and at right angles.
iv A quadrilateral with diagonals which bisect each other.
v A quadrilateral with one diagonal bisected by the other.
vi A parallelogram in which the diagonals are equal.
vii A parallelogram in which the diagonals are at right angles.
viii A parallelogram in which the diagonals are not equal.
ix A quadrilateral in which one pair of opposite sides are equal and parallel.

x A parallelogram with two adjacent sides equal and diagonals equal.

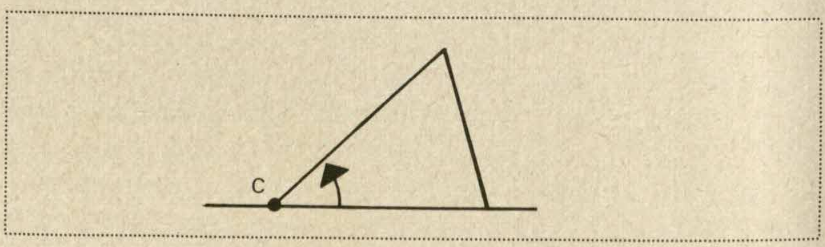
Exercise 6 Here is a drawing of a triangle. Lay a pencil along the side we have marked by a line and rotate it about A until it lies along a second side.



Now rotate it about B in the direction of the arrow.

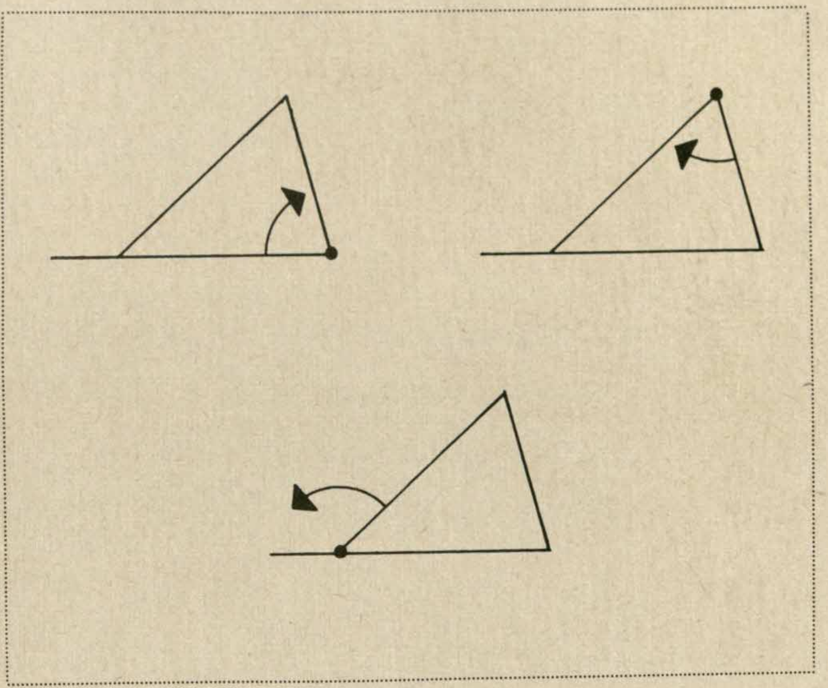


Lastly rotate it about C in the direction of the arrow.



You have now turned your pencil through each of the angles of the triangle in turn. What has happened to the *direction* of your pencil? Would you say that this tells you something about the sum of the angles in the triangle? Express this sum in (i) rotations, (ii) right angles, (iii) degrees.

Exercise 7 Here is a diagram of a triangle with one side produced. Rotate your pencil this time following the three diagrams below.



Has your pencil made a complete rotation?
What can you conclude?
Mark your result on a sketch showing an angle as x, another as y and a third as $x+y$.

Exercise 8 **a** Use the method of the last section to find the sum of the angles in a hexagon, and a dodecagon (12-sided polygon).

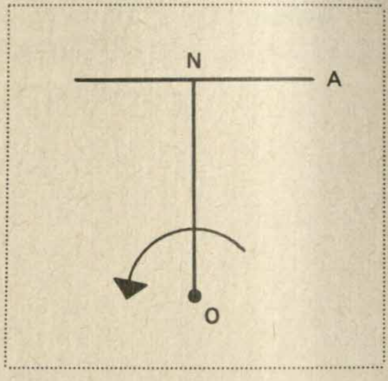
b If each of the two figures above were regular, what would be the size of each interior angle?

c What is the sum of the interior angles of (i) an octagon, (ii) a pentagon?

d If each of the figures in (c) were regular, what sizes would the interior angles be?

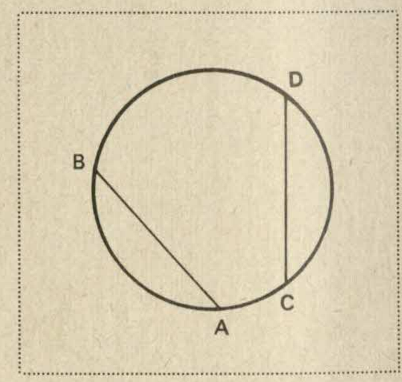
Exercise 9 **a** What is the locus of a point which moves so that it is always the same distance from another point which is fixed?

b What path is traced out by the point A when this figure makes one rotation about the point O?

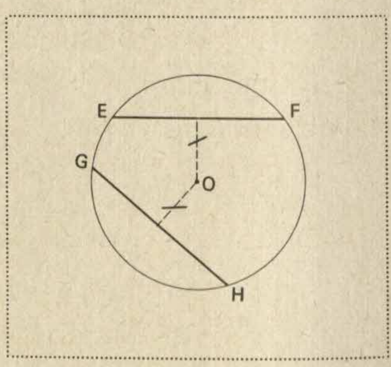


c What path is traced out by the point N in this rotation?

d In the diagram, AB is a chord of a circle and CD is another chord equal to it. What can you say about their respective distances from the centre O of the circle?



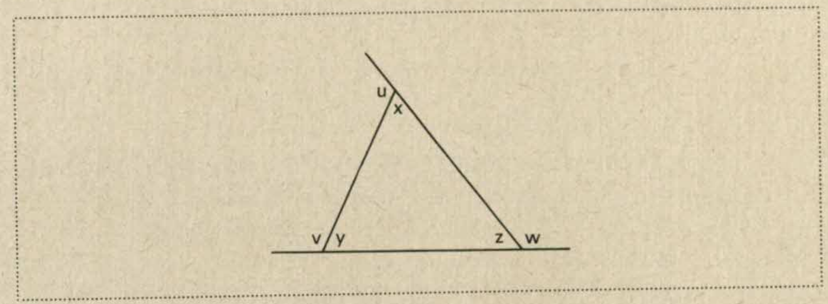
e In this diagram, EF and GH are chords. They are the same distance from the centre of the circle. What do you know about them?



Exercise 10 **a** On a journey a motor car goes round twelve right-angle corners. They are all turns to the right. If the car faced due north when it began the journey, which way does it face at the end?

b A girl sets out from home to go to school. She leaves her front door which faces west and goes down the path to the front gate where she turns right. She goes along the road and turns left twice, once right, twice left and then waits for a bus. The bus takes the first turn to the right and then twice left and stops. The girl turns once to the left and twice right into the school drive. All the corners were right angles. Which direction does the school drive *face* out of the gate?

c



Complete the following table which refers to the diagram above and in which all the angles are given in degrees.

	u	*v*	*w*	*x*	*y*	*z*
a				87	51	
b				57		35
c					46	50
d			125	30		
e	132				25	
f		144				27
g	94				41	
h	142	131				
i	100		147			
j		160	92			

Section 30 **Sweet-shop arithmetic**

Exercise 1 The sweet shop on the corner sells toffee at 5d. an ounce: $\begin{pmatrix}5\\ \\ \end{pmatrix}$. Humbugs cost 3d. an ounce: $\begin{pmatrix}5\\3\\ \end{pmatrix}$. Peppermints cost 4d. an ounce: $\begin{pmatrix}5\\3\\4\end{pmatrix}$.

Joan went to the shop the day before yesterday and bought 2 oz. of toffee: $\begin{pmatrix}2 & & \end{pmatrix}$. She bought also 4 oz. of humbugs: $\begin{pmatrix}2 & 4 & \end{pmatrix}$. At the same time she got 3 oz. of peppermints: $\begin{pmatrix}2 & 4 & 3\end{pmatrix}$.
Keith bought 6 oz. of toffee: $\begin{pmatrix}6 & & \end{pmatrix}$; 8 oz. of humbugs: $\begin{pmatrix}6 & 8 & \end{pmatrix}$, and 2 oz. of peppermints: $\begin{pmatrix}6 & 8 & 2\end{pmatrix}$. Laura bought: $\begin{pmatrix}4 & 6 & 1\end{pmatrix}$.

a i How much money (in pence) did Joan spend?

ii What is $\begin{pmatrix}2 & 4 & 3\end{pmatrix}\times\begin{pmatrix}5\\3\\4\end{pmatrix}$?

b i How much did Keith spend (pence)?

ii What is $\begin{pmatrix}6 & 8 & 2\end{pmatrix}\times\begin{pmatrix}5\\3\\4\end{pmatrix}$?

c i How much did Laura spend?

ii What is $\begin{pmatrix}4 & 6 & 1\end{pmatrix}\times\begin{pmatrix}5\\3\\4\end{pmatrix}$?

d i How much did the shopkeeper take in pence from each child?

ii What is $\begin{pmatrix}2 & 4 & 3\\6 & 8 & 2\\4 & 6 & 1\end{pmatrix}\times\begin{pmatrix}5\\3\\4\end{pmatrix}=\begin{pmatrix} \\ \\ \end{pmatrix}$?

e When they got back to school, Joan, Keith and Laura each wrote down what they had spent in the notation we have been using. They tried to show how they did their calculations too. Joan wrote:

$$(2\quad 4\quad 3)\times\begin{pmatrix}5\\3\\4\end{pmatrix}=2\times 5+4\times 3+3\times 4=34.$$

What did Keith and Laura write?

f They then wrote down what the shopkeeper had taken. Keith began with

$$\begin{pmatrix}2&4&3\\6&8&2\\4&6&1\end{pmatrix}\times\begin{pmatrix}5\\3\\4\end{pmatrix}=\begin{pmatrix}34\\62\\42\end{pmatrix}.$$

So the shopkeeper took . . . . . . pence.
Was Keith right? What is the total? Put in arrows to show how the multiplication was done.

Exercise 2 **a** On the same day Peter and Paul went to the shop. Peter bought 2 oz. of toffee and 3 oz. of humbugs: (2 3). Paul bought 4 oz. of toffee and 1 oz. of humbugs. The prices were still the same: $\begin{pmatrix}5\\ \\3\end{pmatrix}$.
Use the notation to write down Paul's purchase.

b Use the notation to write down the purchases of both the boys (i.e., four numbers in a square.)

c Use the notation to show the multiplication. (You need not show the arrows.) Carry out the multiplication to show the amounts taken by the shopkeeper from each boy.

An arrangement of numbers such as you have been using is called a *matrix*. $\begin{pmatrix}1&7\\4&5\end{pmatrix}$ is a matrix; so is (5 6 7); so also $\begin{pmatrix}8&9&4\\3&6&0\\4&2&7\end{pmatrix}$ and (4 5 9 2 3). We identify different kinds of matrices by the number of rows and columns. For instance, $\begin{pmatrix}9&3&4\\3&6&7\end{pmatrix}$ is a 2-row, 3-column matrix, usually referred to as a 2 by 3 matrix. On the other hand (4 5 6 7) is a 1 by 4 matrix. You have been multiplying matrices, so matrices have at least some of the properties which you normally associate with numbers; for one thing at least, you can multiply them. Later, we shall find out if you can also add them.

Exercise 3 **a** Complete the following by filling in the numbers in the matrix on the RHS.

i $\begin{pmatrix}2&4\\3&5\end{pmatrix}\times\begin{pmatrix}6\\7\end{pmatrix}=\begin{pmatrix}\ \\ \ \end{pmatrix}$ iii $\begin{pmatrix}5&6\\9&8\end{pmatrix}\times\begin{pmatrix}5\\4\end{pmatrix}=\begin{pmatrix}\ \\ \ \end{pmatrix}$

ii $\begin{pmatrix}8&3\\2&1\end{pmatrix}\times\begin{pmatrix}7\\2\end{pmatrix}=\begin{pmatrix}\ \\ \ \end{pmatrix}$ iv $\begin{pmatrix}1&9\\0&4\end{pmatrix}\times\begin{pmatrix}6\\1\end{pmatrix}=\begin{pmatrix}\ \\ \ \end{pmatrix}$

b Work out the following multiplications (do not add up the elements in the final matrix. We did so in the first case for a very special reason. We do not usually need to do so.)

i $\begin{pmatrix}3&4&5\\6&5&3\\9&8&7\end{pmatrix}\times\begin{pmatrix}6\\4\\1\end{pmatrix}$ iii $\begin{pmatrix}5&7\\2&4\\7&8\end{pmatrix}\times\begin{pmatrix}4\\2\end{pmatrix}$

ii $\begin{pmatrix}6&5\\3&8\end{pmatrix}\times\begin{pmatrix}7\\9\end{pmatrix}$

c Invent some problems in purchases (not necessarily sweets) for the matrices of part (b).

We will go back to the purchases made by Joan, Keith and Laura. Joan went into the shop again yesterday. She bought 3 oz. of toffee and 5 of humbugs, and none of peppermints. So on the two days she had spent:

$(2 \quad 4 \quad 3)+(3 \quad 5 \quad 0)=(5 \quad 9 \quad 3).$

Keith did not like the humbugs he had had the day before, so he bought 3 oz. of toffee, none of humbugs and 3 oz. of peppermints. Laura liked all three but she had not much money left. She only bought $(2 \quad 1 \quad 3)$.

Exercise 4 **a** Write down matrices for yesterday's purchases by Keith and Laura.

b Write out the additions for the number of ounces of each sweet that Keith and Laura had purchased on the two days.

c Can matrices be added?

Suppose someone knew about the idea of using frames or letters for unstated numbers but did not understand matrix multiplication and addition. In the next exercise you will be explaining to such a person how matrix processes are done.

Exercise 5 **a** Explain by writing out the answer how this multiplication is done:

$$(a \quad b \quad c) \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \qquad .$$

b Do the same with:

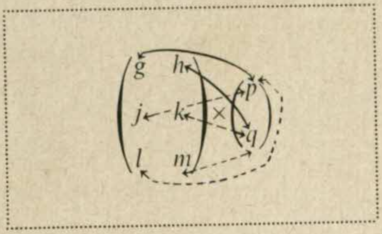
$$(x \quad y \quad z) \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \qquad .$$

c Try with frames instead:

$$(\triangle \quad \square \quad \bigcirc) \times \begin{pmatrix} \triangledown \\ \bigcirc \\ \text{⏢} \end{pmatrix} =$$

d $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} =$

e Do you think this diagram shows the way that it should be done?



Exercise 6 **a** Fill in the frames in the following:

i $\begin{pmatrix} 2 & \triangle \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} \square \\ 6 \end{pmatrix} = \begin{pmatrix} 10+24 \\ 15+6 \end{pmatrix}$

ii $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \bigcirc \\ \triangledown \end{pmatrix} = \begin{pmatrix} 1+1 \\ \frac{1}{2}+2 \end{pmatrix}$

b Find x and y:

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1+x \\ y \end{pmatrix}.$$

Mary and Susan also buy sweets at the corner shop. Every day Mary buys 2 oz. of toffee and 3 oz. of humbugs, while Susan buys 3 oz. of toffee and 2 of humbugs. Yesterday the prices of toffee and humbugs were $\begin{pmatrix}5\\3\end{pmatrix}$ but today new higher prices apply. They are $\begin{pmatrix}6\\4\end{pmatrix}$.

Exercise 7 **a** Write down the matrix multiplication for Mary's and Susan's purchases made yesterday.

b Write out the matrix multiplications for today's purchases.

c Could we express the whole calculation like this:

$$\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix} \times \begin{pmatrix}5 & 6\\3 & 4\end{pmatrix} = \begin{pmatrix}19 & 24\\21 & 26\end{pmatrix}?$$

d Carry out the multiplication of the following:

$$\begin{pmatrix}1 & 6\\5 & 7\end{pmatrix} \times \begin{pmatrix}3 & 2\\2 & 1\end{pmatrix}$$

e Make up a problem of which this could be the solution.

f Simplify:

i $\begin{pmatrix}2 & 7\\5 & 6\end{pmatrix} \times \begin{pmatrix}1 & 9\\4 & 5\end{pmatrix}$ iii $\begin{pmatrix}-4 & 5\\1 & -1\end{pmatrix} \times \begin{pmatrix}2 & -3\\7 & -8\end{pmatrix}$

ii $\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix} \times \begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}$ iv $\begin{pmatrix}7 & -7\\-1 & 1\end{pmatrix} \times \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$

g Take each of the four pairs of matrices in (f) and add them.

h i Multiply the following matrices:

$$\begin{pmatrix}3 & 2\\2 & 1\end{pmatrix} \times \begin{pmatrix}1 & 6\\5 & 7\end{pmatrix}$$

ii Now compare the questions and the answers of this calculation with (d) above. What do you notice?

iii, iv, v Reverse the multiplications of (f) (i), (iii), (iv) and compare the results.

Do you think that matrices are commutative under the operation multiplication? Would f (ii) be commutative? Generally speaking we can see that multiplication is not a commutative operation for matrices. Can they form a number system? There is of course no reason to suppose that there are not particular matrices which are commutative, but we can see at least some which are not. Do you think that matrices are commutative for the operation of addition? Look at some of your additions of matrices and see if they could be reversed.

What do you think all of this implies for the distributive laws? This is a point which you can easily investigate for yourself by taking special cases. You could for instance discover whether

$$\begin{pmatrix}2 & 3\\7 & 6\end{pmatrix} \times \left[\begin{pmatrix}4 & 5\\5 & 4\end{pmatrix} + \begin{pmatrix}6 & 7\\3 & 2\end{pmatrix}\right] = \begin{pmatrix}2 & 3\\7 & 6\end{pmatrix} \times \begin{pmatrix}4 & 5\\5 & 4\end{pmatrix} + \begin{pmatrix}2 & 3\\7 & 6\end{pmatrix} \times \begin{pmatrix}6 & 7\\3 & 2\end{pmatrix}$$

and you could also try putting the first matrix after the bracket to discover what the identity should look like.

Exercise 8 **a**
- i If $7 + x = 7$, what is the value of x?
- ii Fill in the frame $\square$ in the following: $\nabla + \square = \nabla$.
- iii How could you describe to someone, who understood numbers and addition but did not know about 0, what zero does?
- iv What happens when you (i) add on zero and (ii) multiply by zero?
- v Are you ever allowed to divide by zero? Explain your answer.

b
- i If $7x = 7$, what is x?
- ii Fill in the frame $\triangle$ in the following: $\square \times \triangle = \square$.
- iii What is the important property of the number 1?
- iv What happens if you multiply by 1?
- v Are you allowed to divide by 1? If so what result do you get?

c
- i What result do you get if you add $+3$ to -3?
- ii What is the result of $(+94)+(-94)$?
- iii What happens if you add any positive integer and the corresponding negative integer?
- iv What is the value of $-(-6)$?
- v Find $-(-73)$.
- vi What is the negative of a negative integer?

Exercise 9 If two quantities have the property that when you add them the result is zero they are said to be negatives of each other. If $x+y = 0$, then x is the negative of y and y is the negative of x.

a What is the product of 7 and $\frac{1}{7}$?

b What is the value of $x \times \dfrac{1}{x}$?

'One over a number' is called the reciprocal of the number. So $\frac{1}{3}$ is the reciprocal of 3. What is the reciprocal of $\frac{1}{3}$?

If two quantities have the property that when you multiply them the result is 1, they are said to be reciprocals of one another. 'Negative' and 'reciprocal' are examples of what we call 'inverses'. The negative of a number is its inverse with respect to addition, while the reciprocal of a number is its inverse with respect to multiplication. It is important that all inverses obey the commutative law. This will be particularly important in this section.

Exercise 10 In this exercise capital letters will be used to stand for matrices.

$$X = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad Y = \begin{pmatrix} 4 & -1 \\ -2 & 7 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

a Find (i) AX, (ii) XA, (iii) BY, (iv) YB, (v) XC.

b Work out (i) A + X, (ii) Y + B, (iii) X + C, (iv) Y + D.

c Experiment with multiplication of both X and Y by any of the matrices A to J to discover whether any of them is (i) the unit matrix (i.e. it behaves like the ordinary real number 1) and (ii) the zero matrix (which should behave like 0). Do not forget that such matrices must be commutative to be zero and unity elements.

d If you find the matrix which appears to be the zero matrix, try adding it to another matrix (and adding another matrix to it) to see if it satisfies the requirements which we know of zero.

e See if you can find the matrix which is the inverse for addition (i.e., the negative) of: (i) X and (ii) Y.

f Multiply X by the matrix:

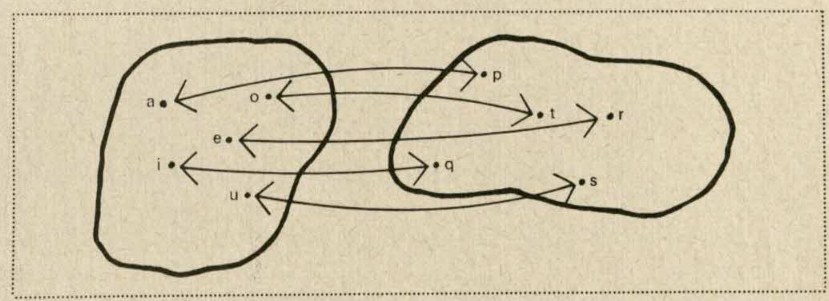
$$K = \begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

g What can you say about the relationship of K and X? Is it commutative?

Section 31 **Mappings**

Some time ago we used mappings to see the ways sets were like each other. We used one-to-one mappings. Here are some examples:

$$\begin{array}{rcccl} A = \{ & a, & b, & c & \} \\ & \updownarrow & \updownarrow & \updownarrow & \\ B = \{ & \alpha, & \beta, & \gamma & \} \end{array}$$

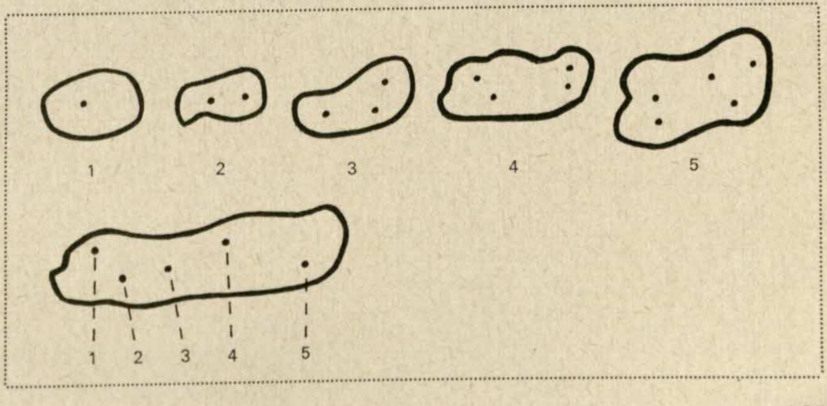


$E = \{\bigcirc, \diamond, \square\}$ $F = \{\flat, \sharp, \natural\}$

Exercise 1 **a** What can you say about the sets A and B? Can you say the same about any of the other pairs of sets in the example?

b Explain what we mean by the *number of a set.*

c We have two ways of using numbers: *cardinal* numbers are the numbers referred to in 1(b); the others are *ordinal* numbers. Explain what we mean by ordinal numbers. Here is a diagram to help you.



Exercise 2 **a** Here is a mapping of two sets in one-to-one correspondence.

$\{x, y\}$ $\{\text{I}, \text{II}\}$

Is this the only way a one-to-one mapping can be done for this pair of sets? How many different ways can it be done in fact?

b Find out how many ways you can set up one-to-one mappings for pairs of sets whose cardinal numbers are (i) 3, (ii) 4, (iii) 5. You will find that large diagrams and coloured lines will help you.

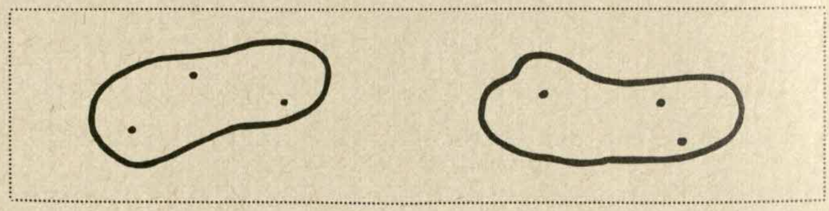
c We have begun a mapping of two infinite sets:

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\}$
↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕
$\{\text{I}, \text{II}, \text{III}, \text{IV}, \text{V}, \text{VI}, \text{VII}, \text{VIII}, \text{IX}, \ldots\}$

Can this mapping be continued for all the elements of the two sets? Could you actually do it for all the elements? Do you need to in order to know that it can be done?

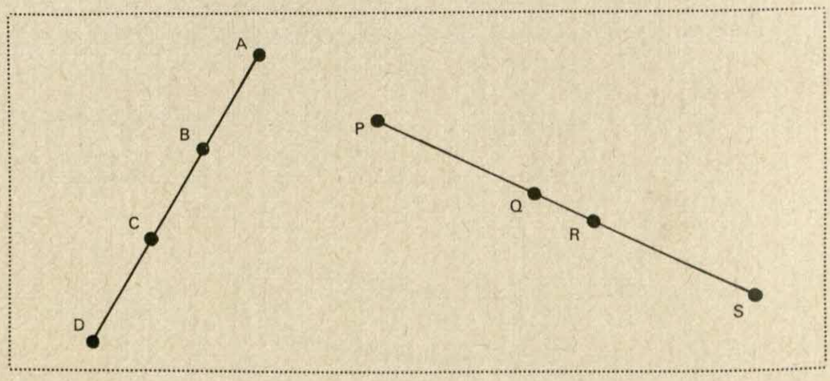
d What could you say about the numbers of elements in the two sets in (c)?

Exercise 3 **a**



Here is a Venn diagram showing two sets of points. Can you illustrate a one-to-one mapping of these two sets?

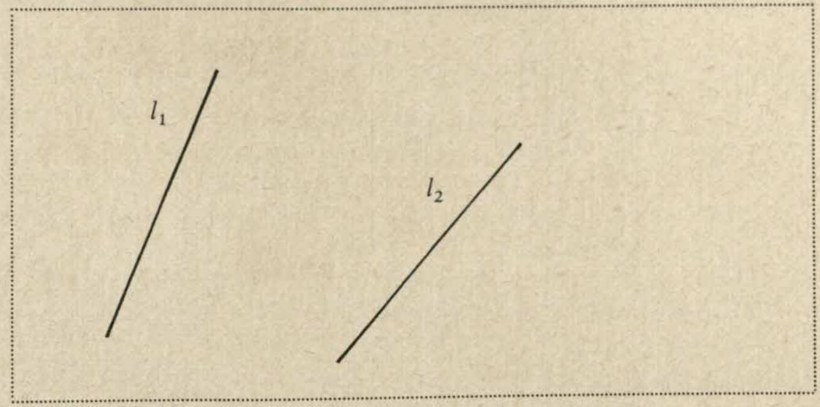
b



This diagram shows two line segments lying in a plane. Can you illustrate a one-to-one mapping of the set of points A, B, C, D into the set P, Q, R, S?

Does your mapping keep the same order in the two sets? Would it be possible to do the mapping a different way?

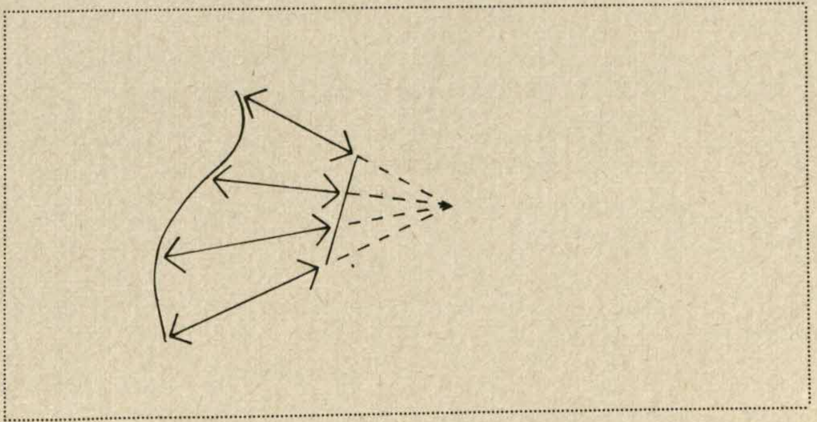
c



Could you find a mapping which sets up a one-to-one correspondence between all the sets of the points in these two segments of straight lines which keeps the same order in each? (They are both the same length.)

d What could you say about the numbers of elements in the two sets l_1 and l_2 in (c)?

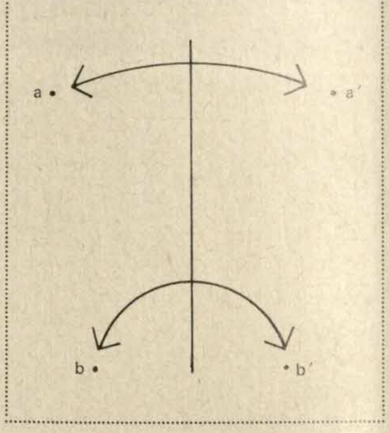
e



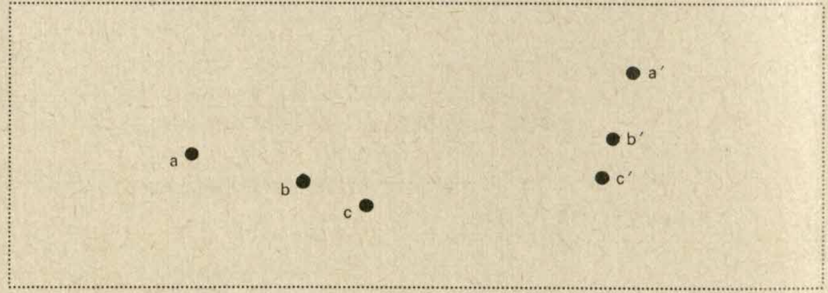
You have seen this diagram before in Book 1. The two line-sets are segments of different kinds of line and they are different lengths. We have shown a few specimen arrows for the mapping of one set into the other. The dotted lines show how we drew them. Could we continue this process indefinitely? What could you say about the numbers of elements in each of the sets?

f Do you think you could map these points into the natural numbers? How many natural numbers are there? Are there more or fewer points in the segment of a line?

In this mapping we say that the set $\{a', b'\}$ is the *image set* of the set $\{a, b\}$. We could equally well say that $\{a, b\}$ is the image set of $\{a', b'\}$. The element a' is the image of element a and b' is the image of element b.



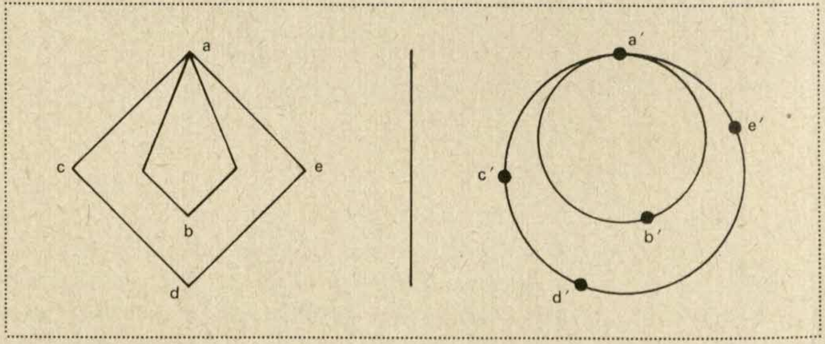
Exercise 4 a



Use the same two sets as in the example above and show a one-to-one mapping which gives the image of a as b'.

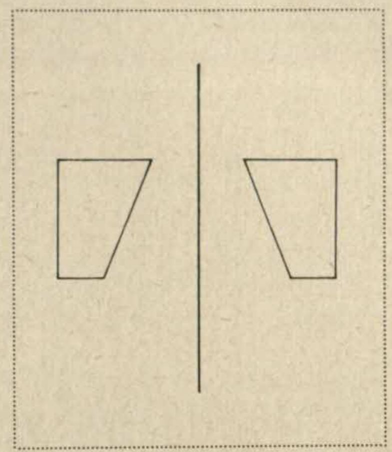
b In these two sets draw arrows to show a one-to-one mapping which gives c' as the image of a and makes a' the image of b.

c

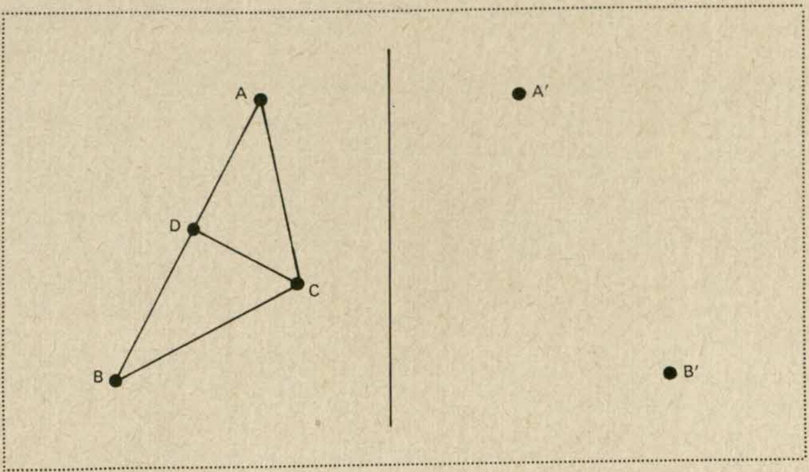


These two sets are separated by a line. Draw arrows to show how a may be mapped into a' and b into b'.

d The *reflection of points in a line* is a very special sort of mapping. Here is an example:



The images of points A and B are A′ and B′. The position of A′ and B′ are found by making the distance of A from the line the same as the distance of A′ from the line, and similarly with B and B′.



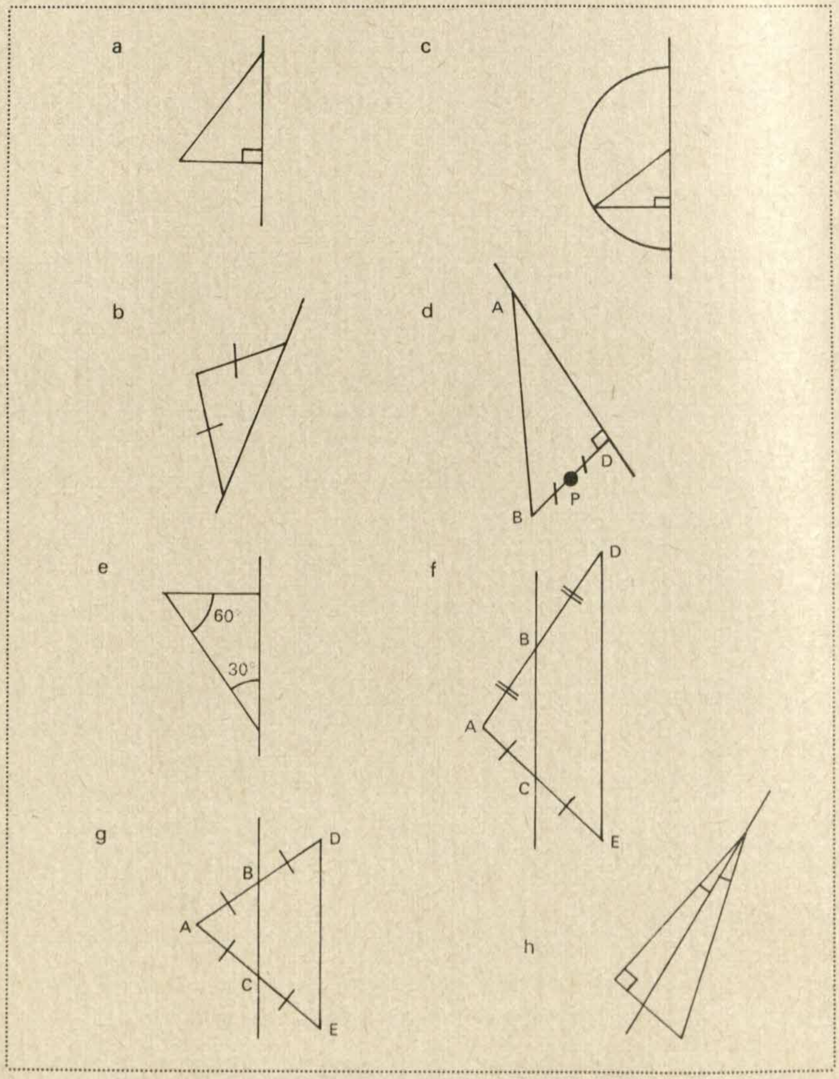
Find the position of C′ and D′ and put them into your copy of the diagram. Now complete the reflection.

e The points of the segment AB are reflected in a line *l*. The image set is A′B′. If C is a point 2 in. from *l*, where is its image? Draw a sketch on lined paper, putting in *l* at right angles to the printed lines. A is 3 in. from *l* and B is 1 in. from *l*. Put in both C and C′.

f In the last diagram, where is the image of the point B′?

g In the same diagram, where would be the image of a point $D \in l$?

Exercise 5 The following sketches show a number of figures and for each a line *l*. Copy the sketches and draw the reflection of the figure in the line in each case.



Mark in your diagrams any equal lines and angles. Find any other special properties of your figures. You may find paper-folding helps in some cases.

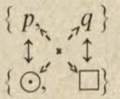
Exercise 6 **a** What word would you use to describe the triangles in Exercise 4(d)?

b Do you think it would be appropriate to use the same work to describe the figures in the other diagrams? Does the word mean 'the same'?

c Are distances *preserved* in reflection?

d Do you think the sort of reflection we are talking about here is like looking into a mirror? Is the triangle of Exercise 5(e) 'looking into a mirror'?

We have just been thinking about one special sort of mapping. Later we shall be finding some more examples in geometry. Now we are going to think once again about sets of elements which are not necessarily geometrical points.
Here we have set $S = \{p, q\}$ and set $T = \{\odot, \square\}$. We shall take one element from each and place them inside parentheses: (,). We shall put the element from set S first. We call the result a couple. A couple we can make this way is $(p, \square)$. We can make a lot more couples. How many altogether? This diagram may help you to see how many there are:



Let us make a new set. Its elements will be the couples we can make from set S and set T. We shall call it V, the product set of S and T. We can write:

$V = S \times T = \{(p, \odot), (p, \square), (q, \odot) (q, \square)\}$

(× is read *cross*)

We will also make a product set $W = T \times S$.

$W = T \times S = \{(\odot, p), (\square, p), (\odot, q), (\square, q)\}$

You will see that the two sets V and W are not the same. $V \neq W$. Is $S \times T = T \times S$?
The reason that the sets V and W are not equal is that *order* counts for the members of a couple. For this reason couples are sometimes called 'ordered pairs'.

Exercise 7 $A = \{a, b\}$, $B = \{\alpha, \beta\}$, $C = \{\triangle, ◫\}$,
$D = \{a, b, c\}$, $E = \{\diamond, \triangle, \square\}$, $F = \{\flat, \sharp, \natural\}$,
$G = \{1, 2, 3\}$, $H = \{5, 6, 7\}$, $J = \{3, 4, 5\}$

a Find the product sets:

i $A \times B$
ii $C \times B$
iii $A \times G$
iv $D \times E$
v $E \times D$
vi $D \times F$
vii $F \times G$
viii $G \times H$
ix $H \times J$
x $J \times G$
xi $G \times G$
xii $J \times J$

b In ordinary numbers $2 \times 3 = 3 \times 2$. This is the *commutative law for multiplication.* Is the commutative law for multiplication true of the products of sets?

You can make up product sets from two sets which are not equivalent:

$$K = \{2, 3\} \qquad L = \{0, 1, 2\} \qquad M = \{0, 1, 2, 3\}$$

$$K \times L = \{(2, 0), (3, 0), (2, 1), (3, 1), (2, 2), (3, 2)\}$$

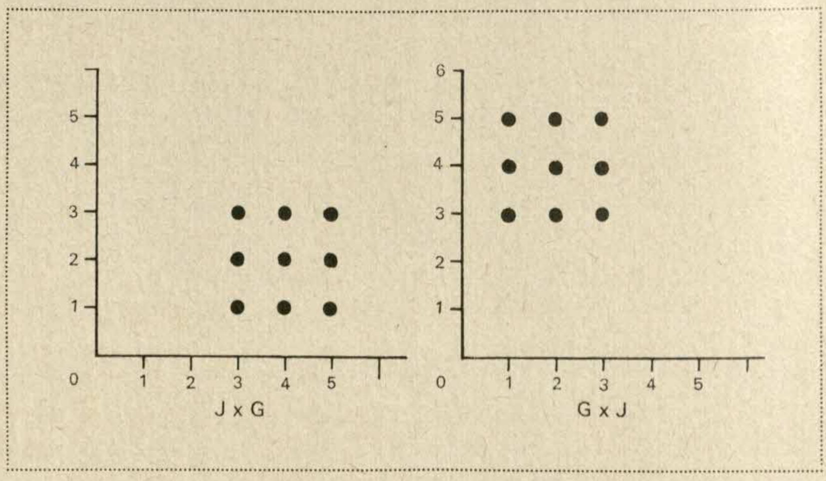
Exercise 8 **a** Find the product sets:

i $G \times K$
ii $K \times H$
iii $J \times L$
iv $K \times M$
v $M \times L$
vi $M \times G$

b Do you think you could make up a product set $A \times B \times C$? Try it.

We can make diagrams of product sets. These diagrams are called *graphs.* We are usually only interested in the products of sets of numbers. The graphs we shall draw for these are something like the graphs of ordinary sets of numbers which we have drawn before.

Exercise 9 **a**



J x G

G x J

Which set of the product set goes 'horizontally'?

b Which number in a couple goes on the 'vertical' axis?

c Copy the graph of $J \times G$ and put in dotted lines to show how each couple is plotted on the graph.

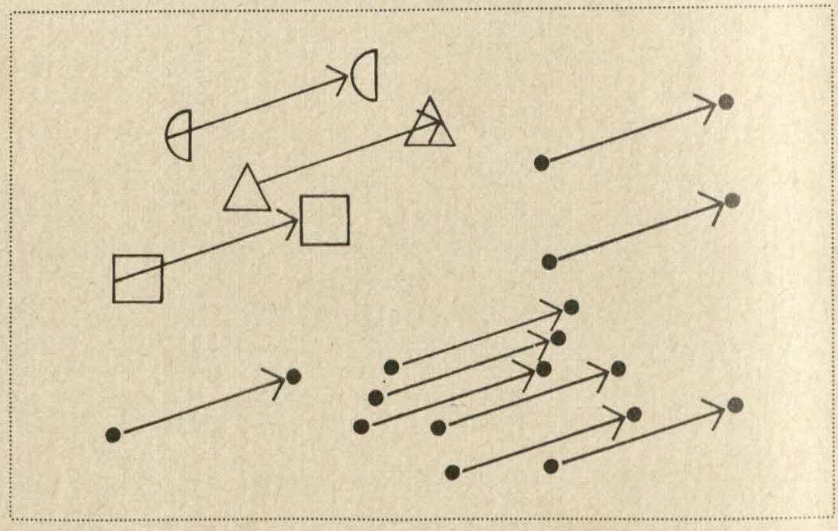
d Draw graphs of

i $G \times H$
ii $H \times J$
iii $G \times G$
iv $K \times L$
v $G \times K$
vi $K \times H$
vii $J \times L$
viii $K \times M$
ix $M \times L$
x $M \times G$

e In the diagrams of J × G and G × J one set of points could be regarded as a reflection of the other if they were together on the same pair of axes. Where is the axis of reflection? Does this work for

i K × L and L × K
ii M × L and L × M
iii any pair of products of this type?

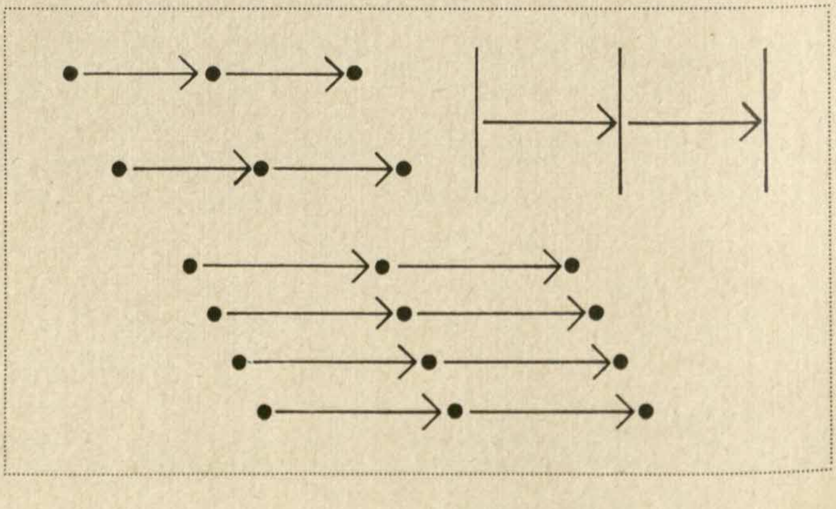
Translation



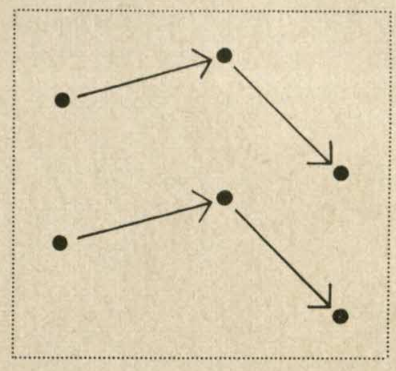
When we move a set of points a given amount in a given direction, we can say that we have *translated* the set of points.

In the above diagram we have shown a translation of several subsets of the plane, all by the same amount in the same direction.

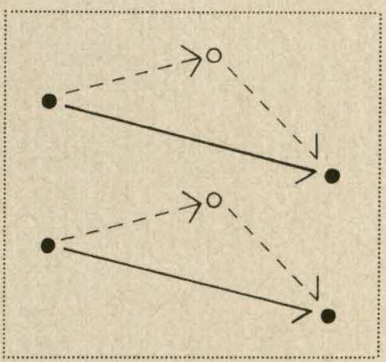
Here is a set of diagrams in which three sets are translated twice.



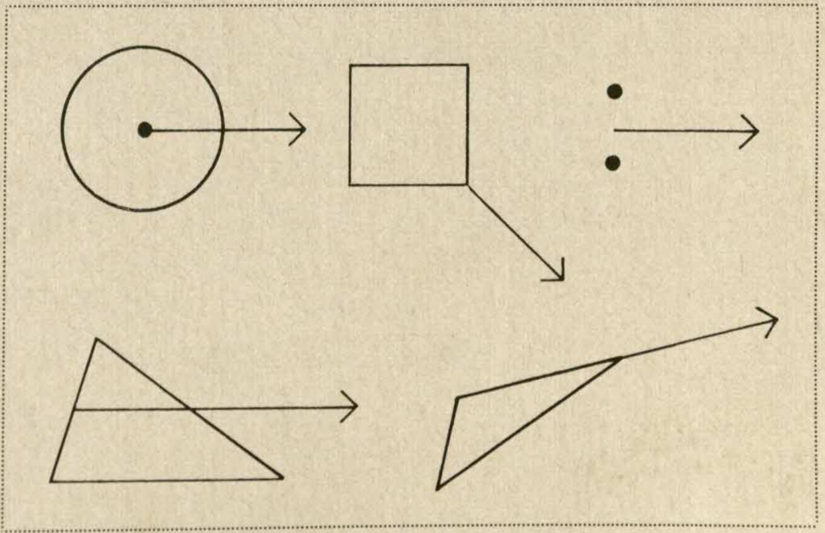
If we translate first in one direction and then another as here:



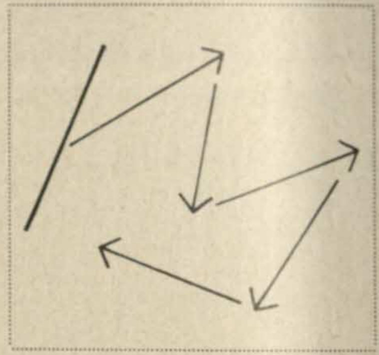
we obtain a result which we could have obtained by a single translation:



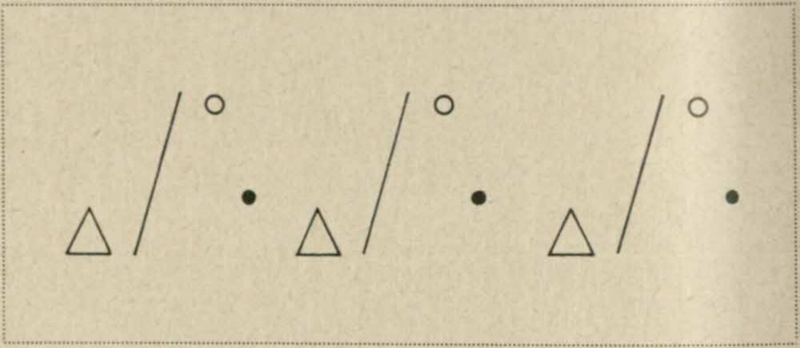
Exercise 1 **a** Translate each of the following sets as shown. Copy the diagrams as carefully as you can but do not worry if your drawings are not exact.



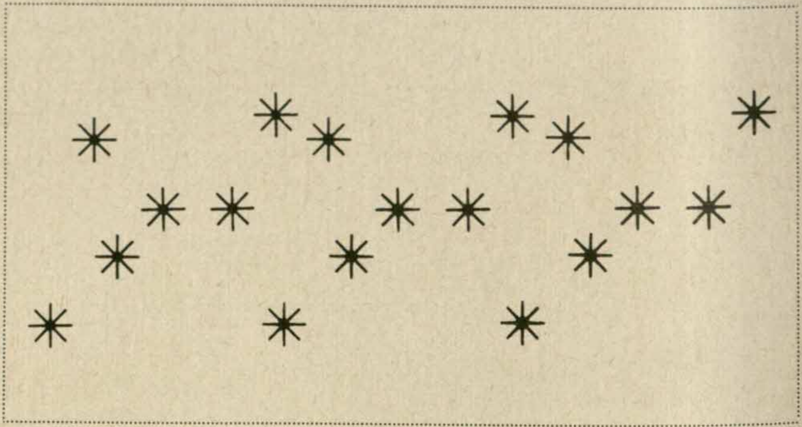
a (*continued*)



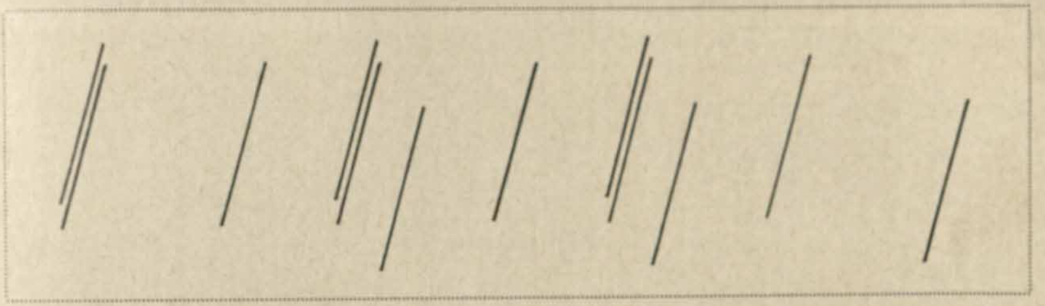
b What is the set which has been translated in this case?



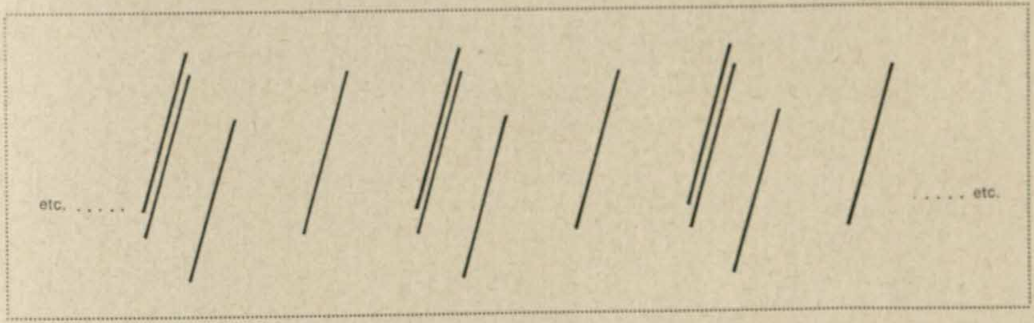
c How far is the set being translated in this *frieze*? (Measure it in the book.)



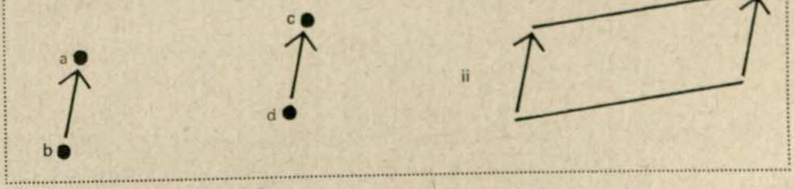
d Here is a set of points which has been translated twice. Which is the set?



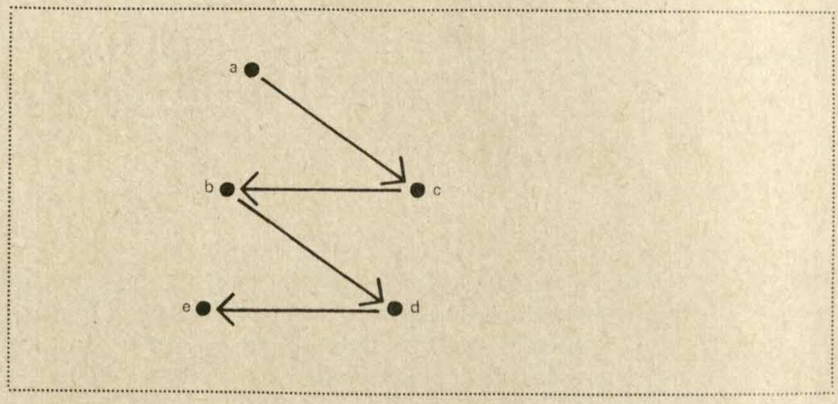
e This diagram is almost the same but represents the set of the previous question translated an infinite number of times in both directions. It is an infinite frieze. Can you see which set has been translated this time? Is there only one answer?



Exercise 2 **a** Look at the two pairs of diagrams (i) and (ii). What do you conclude about the second diagram in each?

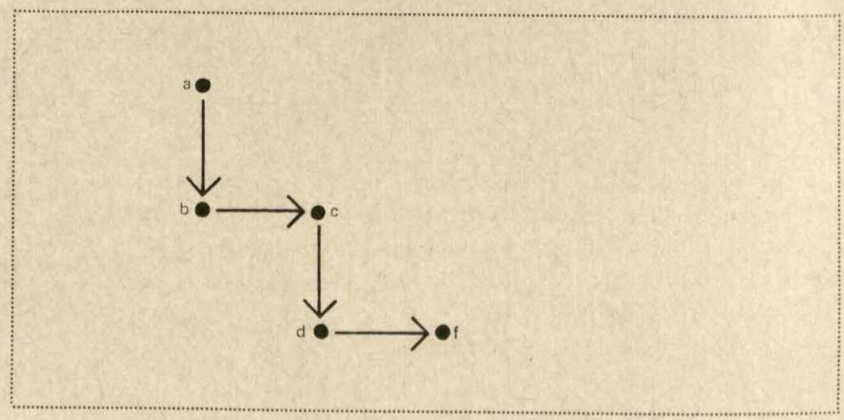


If two pairs of points can be connected in this way by a parallelogram, they are said to be *equipollent*. We can write $(a, b) \uparrow (c, d)$.
((a, b) is called a *couple*.)



b Write down the equipollences you can see in this diagram.

c In (a) above we said $(a, b) \uparrow (c, d)$. Do you think we could equally well have written $(a, c) \uparrow (b, d)$? Why?

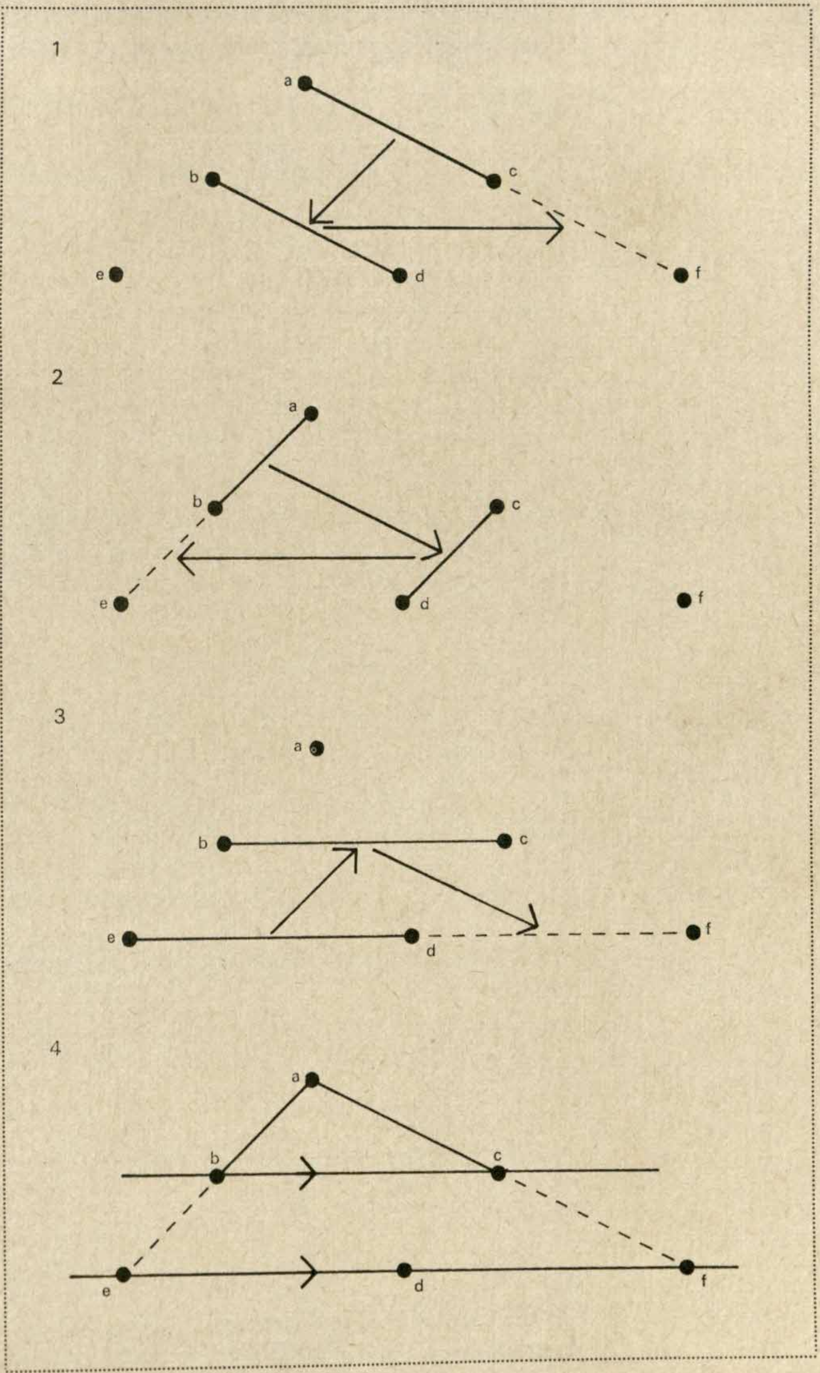


d Look at this diagram and write down the equipollences you can see.

e What can you say about the point b in the line segment ae?

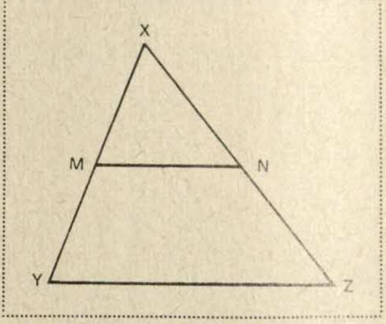
f What can you say about c in the line segment af?

Exercise 3 **a** Look carefully at the following series of drawings. Follow the equipollent couples in each stage and note your results.

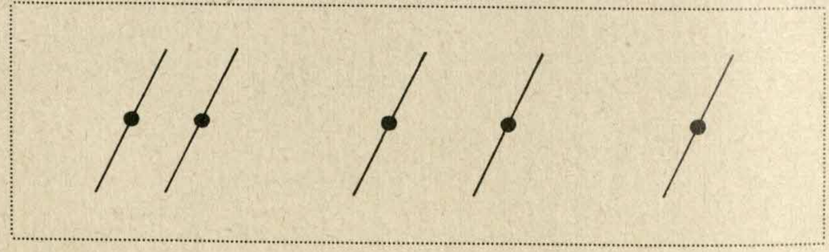

1
a
b
c
d
e
f
2
a
b
c
d
e
f
3
a
b
c
d
e
f
4
a
b
c
d
e
f

b What do you conclude about the line segments *bc* and *ef*? What are their directions and what connexion can you see in their lengths?

The fact which you have just discovered is sometimes called the 'mid-point theorem': M and N are the mid-points of the sides XY and XZ of triangle XYZ, MN is parallel to YZ and length MN = $\frac{1}{2}$ length YZ. This is an interesting and important fact which is true of all triangles.

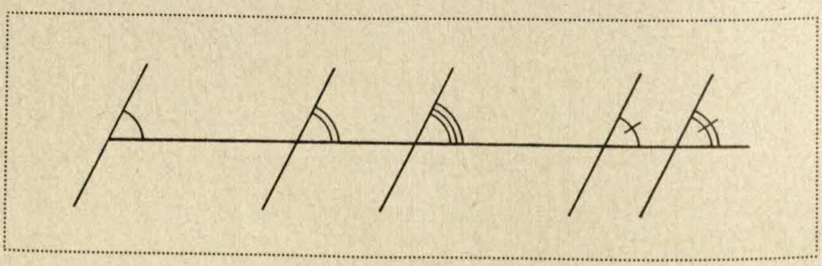


Exercise 4 **a** Here is a set of line segments. Each is a translation of a previous one. All the translations are in the same directions but they are not all translated the same distance. The dot in each represents the motion of a particular point.



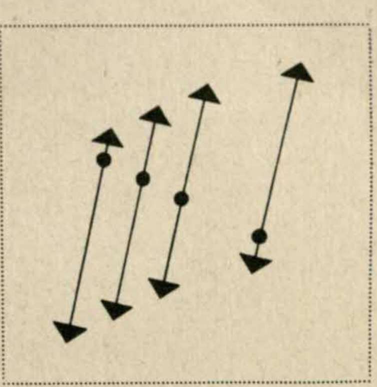
What can you say about the motion of the end points of the segment?

b

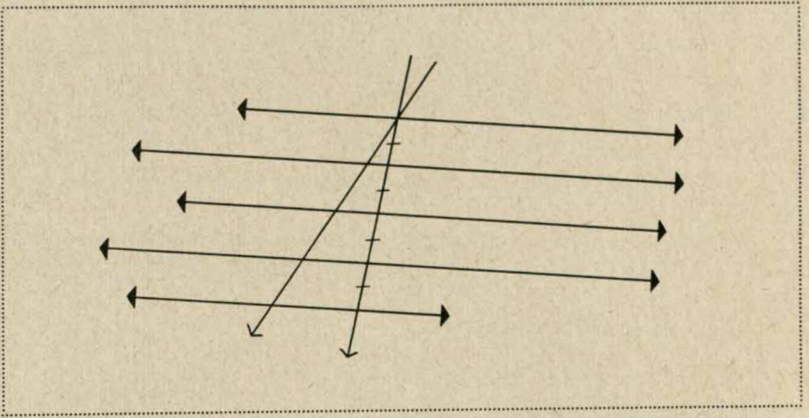


In the diagram of (a) above, what can you say about the angles we have marked?

c Here is a set of *lines*. The dot again indicates the motion of a point. Could we have obtained the same result (i.e. the same positions for the lines) by motion in a different direction? Explain.

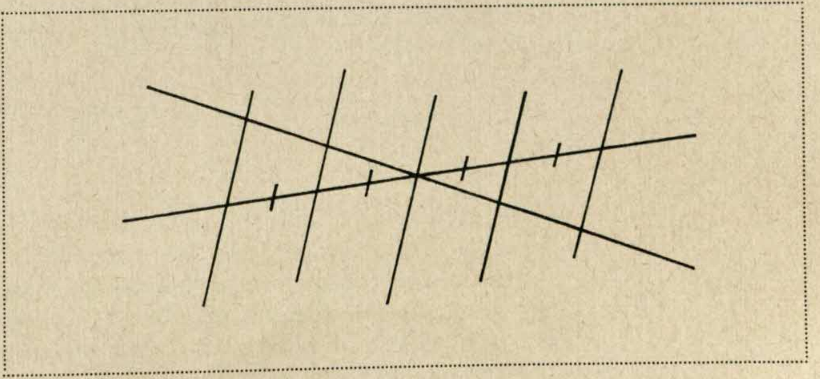


d



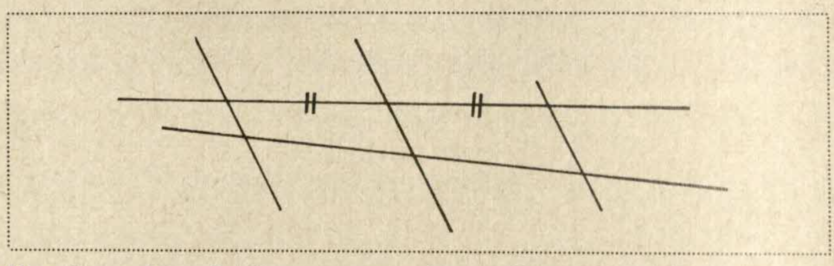
Here is a set of parallel lines and two alternative motions for producing them. Copy the diagram and mark another set of equal segments.

e



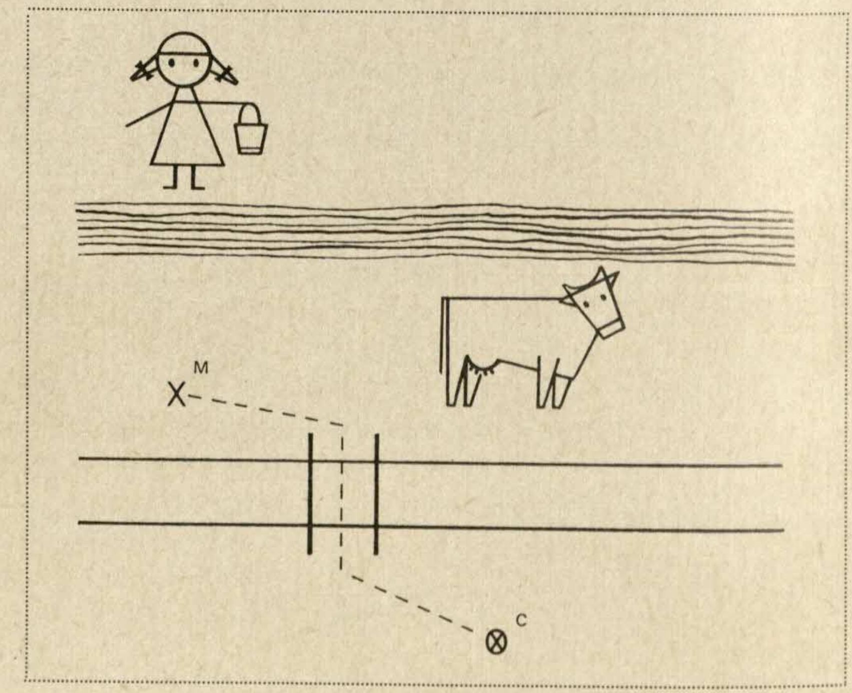
In this diagram mark all the equal segments you can find.

f



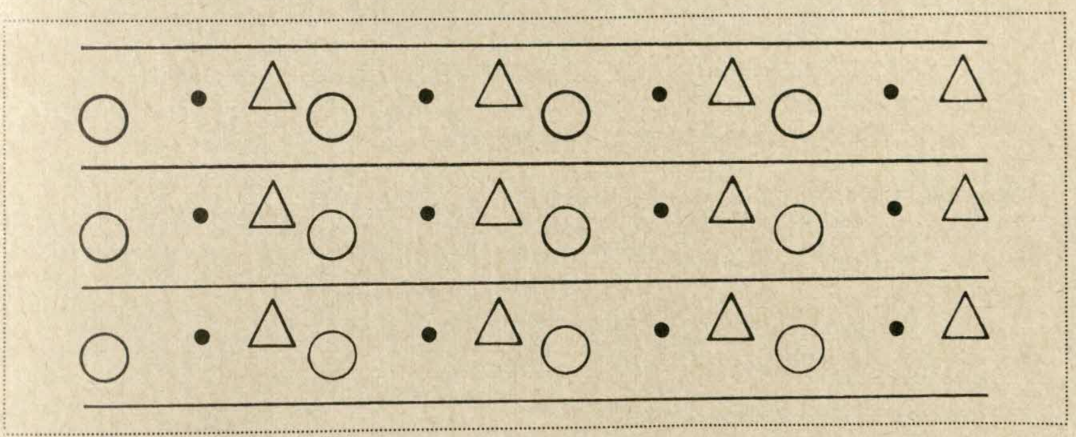
Here is set of parallel line segments with two lines crossing it. (These lines are sometimes called *transversals*.)
Mark any equal segments there are.

Exercise 5 Here is a problem about a milkmaid. It is a famous problem and you can use the idea of translations to solve it.

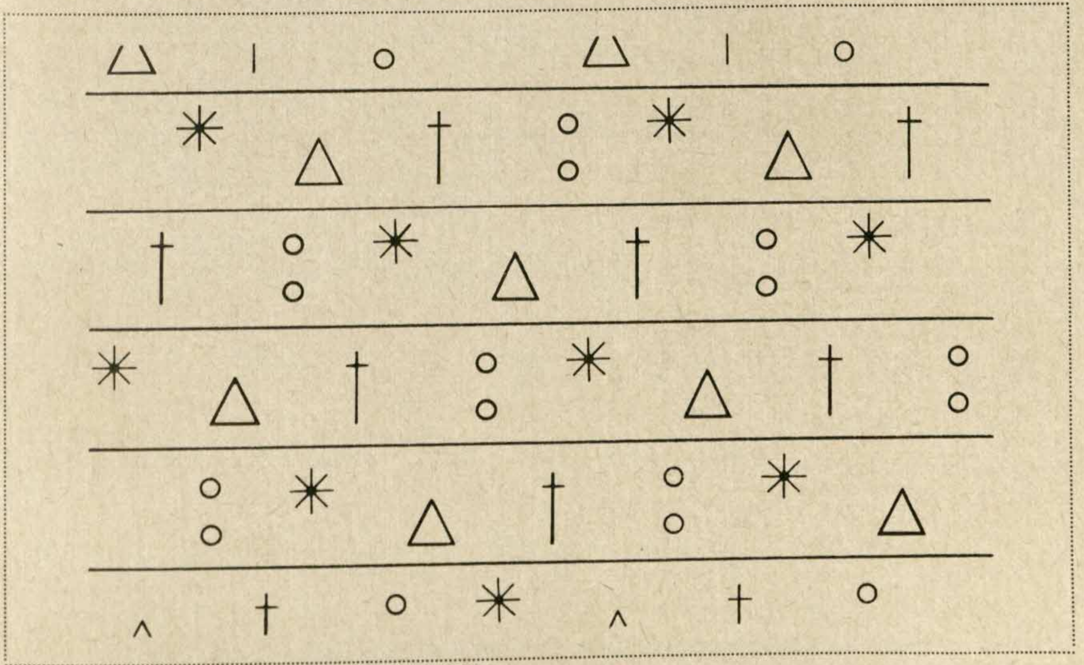


The milkmaid has to milk the cow in the field but she needs a bridge to cross the stream. She naturally does not want to walk any further than she has to. Her problem is where to put the bridge to make the journey as short as possible. Can you find the position for her? (The bridge must also be as short as possible.) (Hint: she has to cross the river in any case, so translate the milkmaid the length of the bridge in the direction which the bridge will be.)

Exercise 6 Here is a practical exercise for you to do in the next few days. Find as many examples of translations as you can around you. Look at floors, ceilings, walls, curtain patterns and book covers for instance. In Book I you did something called tessellation. Was this a translation? Remember a frieze like the ones we have already seen can be translated again to form an overall pattern which can cover the plane entirely.



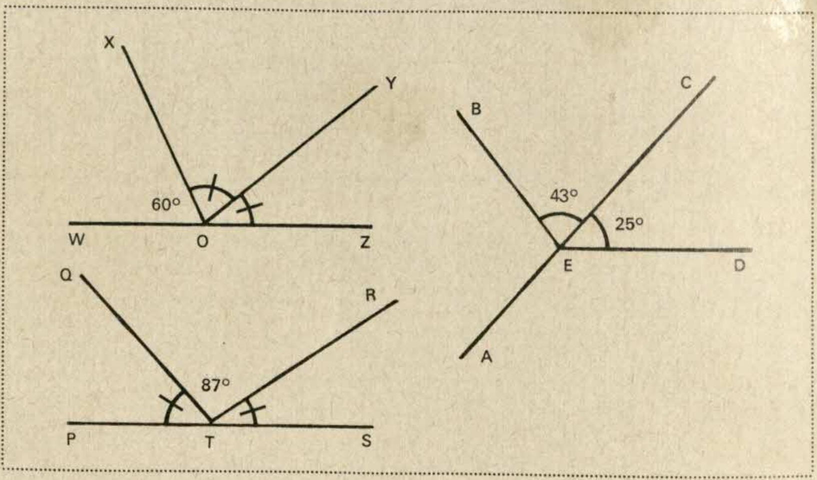
This sort of set of images is called a *tapestry*. Why? Can you find some examples of this too?



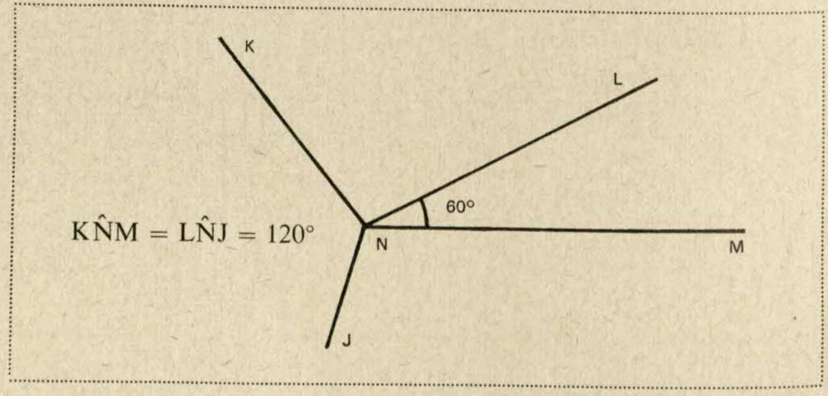
Section 33 **Calculations in geometry**

In each of the following exercises you will need a sketch. Do not draw it accurately but make it good enough to show the information you are given clearly. You will probably find coloured pencils useful for marking angles and lines equal on your diagrams. You must *work out* the answers and not attempt to measure them.

Exercise 1 In these diagrams find $X\hat{O}Y$, $R\hat{T}S$, and $A\hat{E}D$.



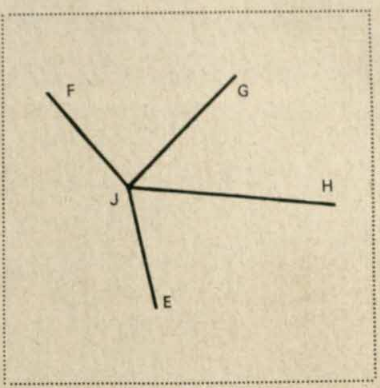
Exercise 2



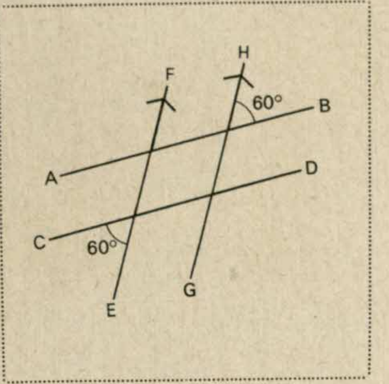
$K\hat{N}M = L\hat{N}J = 120°$

Which three points lie in a straight line? Explain.

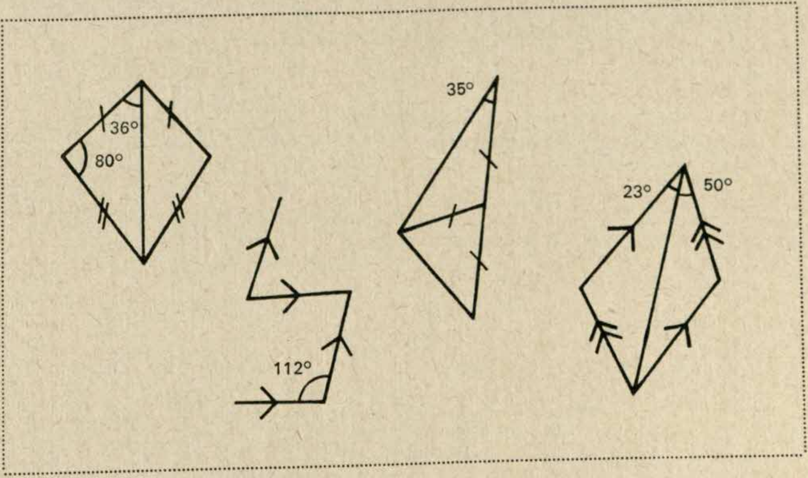
Exercise 3 If $F\hat{J}H = G\hat{J}E$, what can you say about $F\hat{J}G$ and $H\hat{J}E$? Explain.



Exercise 4 What can you say about the lines AB, CD, EF, GH? Find the sizes of all the angles in the figure.

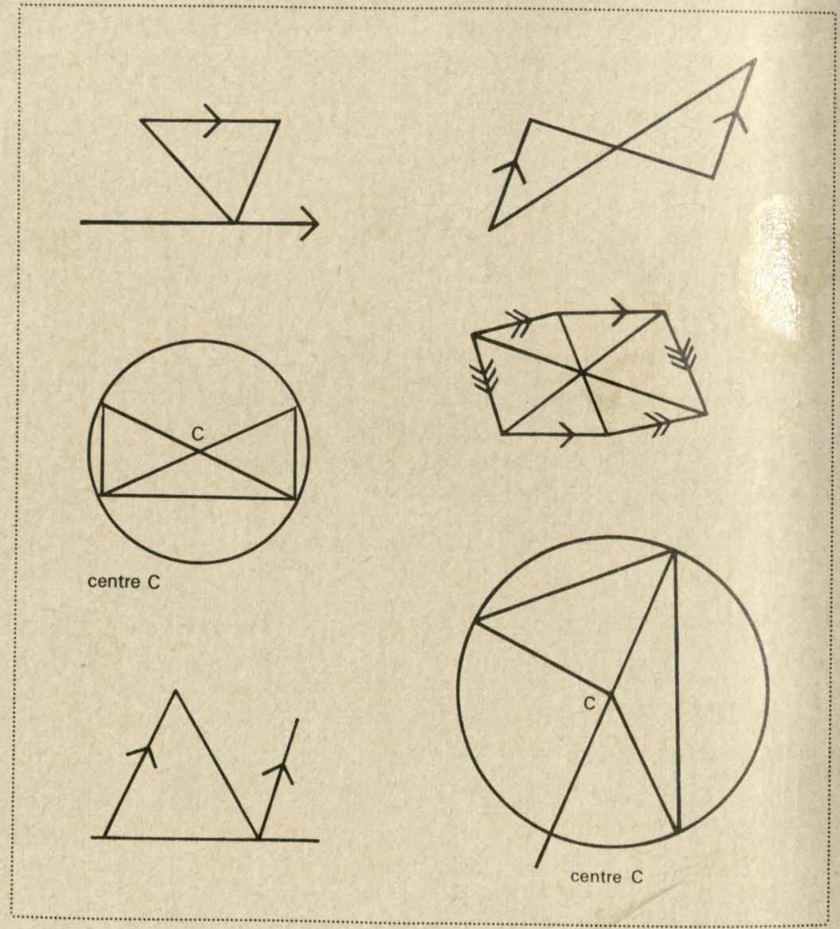


Exercise 5

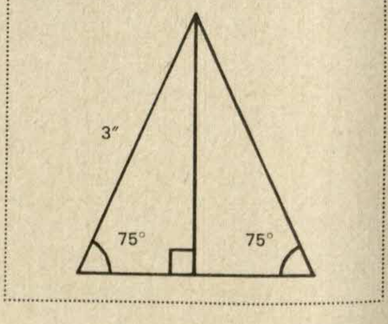


On your diagrams fill in the magnitudes of the other angles.

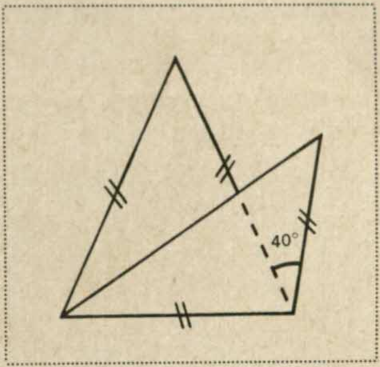
Exercise 6 Mark pairs of equal angles on your diagrams.



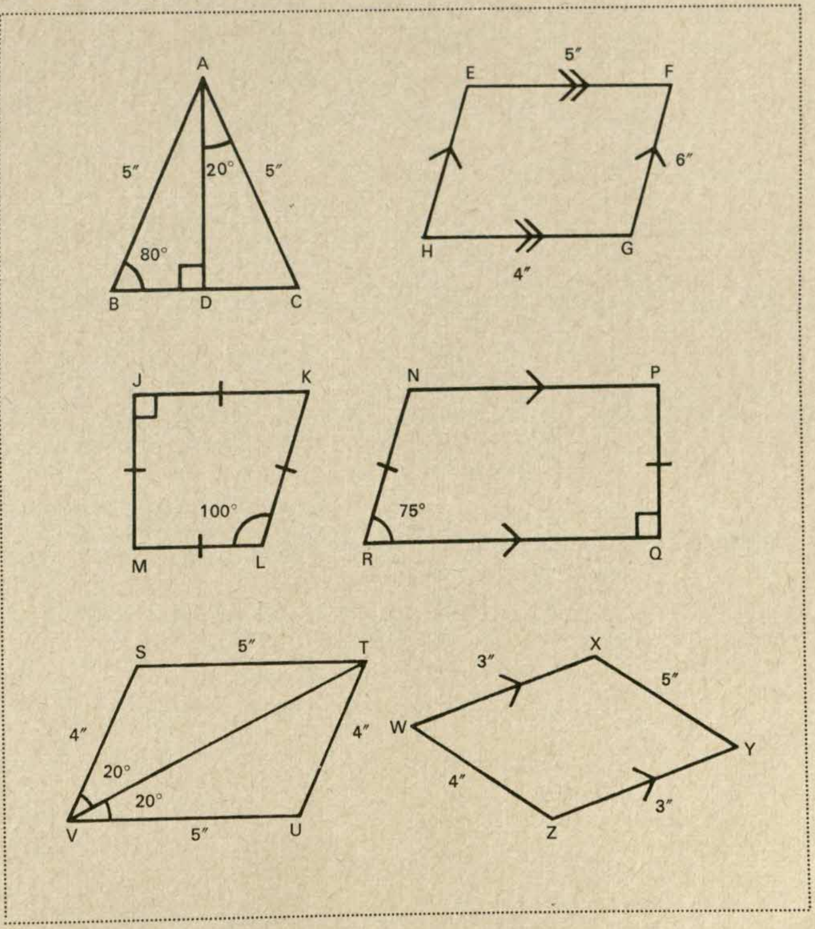
Exercise 7 Give whatever information you can about this figure.



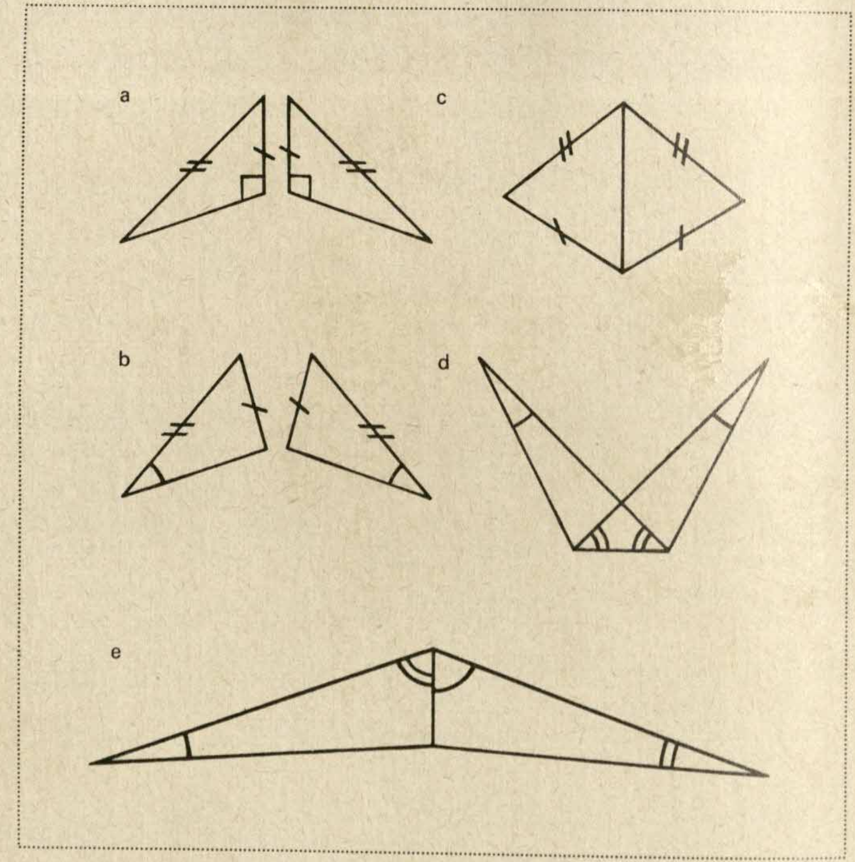
Exercise 8 Find whatever information you can about these triangles.



Exercise 9 What is wrong about each of the following diagrams?



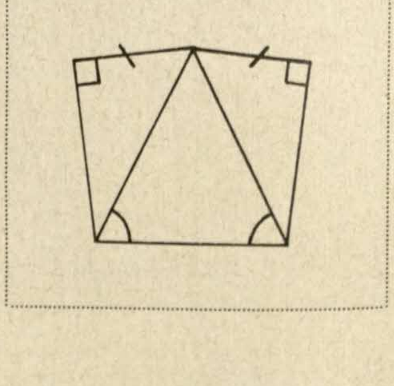
Exercise 10



What can you say for certain from the information given about each pair of triangles?

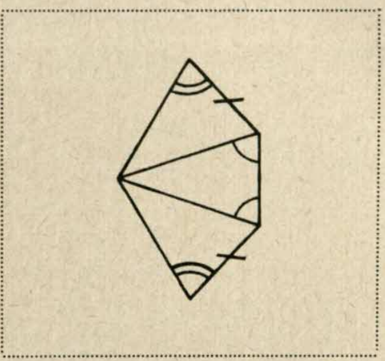
Exercise 11 Can you find two congruent triangles in this figure? How do you know they are congruent?

Is the figure certain to be symmetrical about an axis? If it is, draw in a broken or coloured line to show the axis.

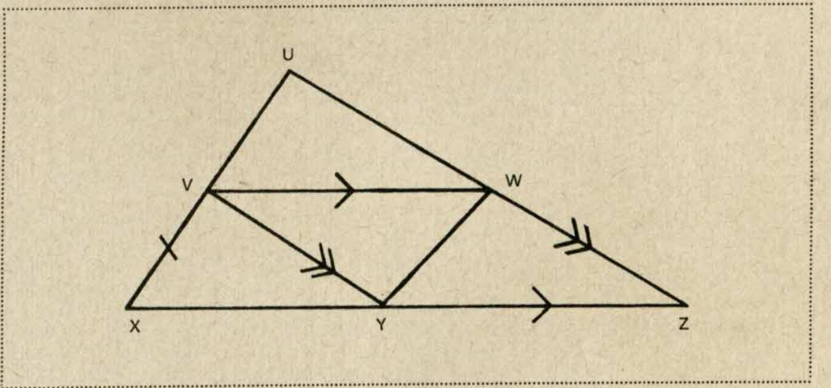


Exercise 12 Can you be certain of two congruent triangles in this figure?

Is the figure certain to be symmetrical? If so draw in the axis as before.

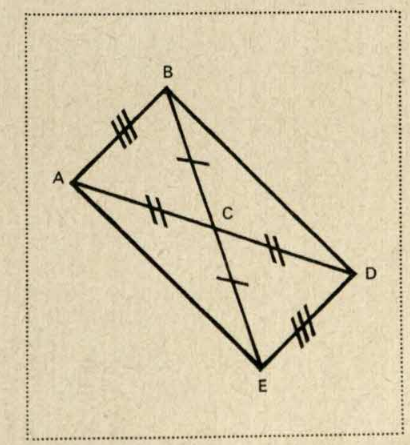


Exercise 13

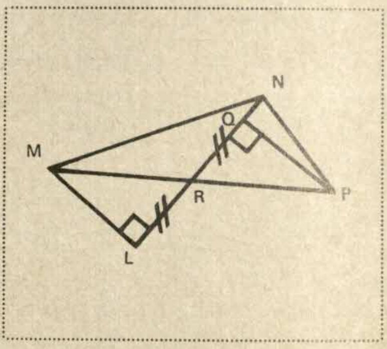


Mark in all the sides and angles which you know are equal. Say why they are in each case.

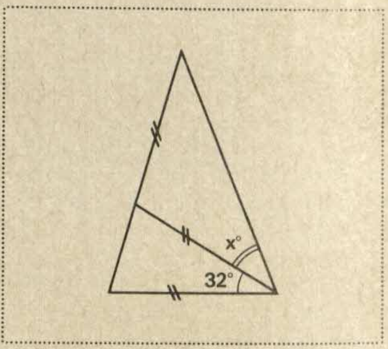
Exercise 14 Write down all the facts you can find about this figure. Give your reasons.



Exercise 15 Can you show that ML = QP?



Exercise 16 Find x:



Exercise 17 XYZ is an acute angled triangle. P is a point inside it. PX = PY = PZ. Now $X\hat{P}Y = 140^\circ$ and $Z\hat{P}X = 130^\circ$. Find $Y\hat{P}Z$ and all the angles of triangle XYZ.

Section 34 **Implication**

Here are some statements. Look at them very carefully to find out what they mean.

$a = b$ and $b = c \Rightarrow a = c$
$a = b$ and $b = 2c \Rightarrow a = 2c$
$a = b$ and $b = 3c \Rightarrow a = 3c$
$a = 3b$ and $b = c \Rightarrow a = 3c$
$a = 3b$ and $b = 2c \Rightarrow a = 3 \times 2c$

$a = 4b$ and $b = 5c \Rightarrow a = 4 \times 5c$
$a = 2b$ and $b = 4c \Rightarrow a = 8c$
$f = 5g$ and $g = 2h \Rightarrow f = 10h$
$x = 10y$ and $y = 5z \Rightarrow x = 50h$
$j = 2km$ and $k = 3p \Rightarrow j = 6pm$

Exercise 1 These statements are not yet complete. Complete them.

a i $a = 3b$ and $b = 3c \Rightarrow a =$
ii $a = 4b$ and $b = 2c \Rightarrow a =$
iii $a = 3b$ and $b = 4c \Rightarrow a =$
iv $a = 7b$ and $b = 3c \Rightarrow a =$
v $b = 3c$ and $c = 2d \Rightarrow b =$
vi $b = 6c$ and $c = 2d \Rightarrow b =$
vii $b = 5c$ and $c = 3d \Rightarrow b =$
viii $b = 4c$ and $c = 2d \Rightarrow b =$
ix $b = 3c$ and $c = 4d \Rightarrow b =$
x $b = 10c$ and $c = 2f \Rightarrow b =$

b i $d = 4f$ and $f = 6g \Rightarrow d =$
ii $d = 2f$ and $f = 5g \Rightarrow d =$
iii $d = 5f$ and $f = 2g \Rightarrow d =$
iv $d = 2f$ and $f = ab \Rightarrow d =$
v $d = af$ and $f = 2b \Rightarrow d =$
vi $d = ag$ and $g = 3c \Rightarrow d =$
vii $d = ak$ and $k = 13h \Rightarrow d =$
viii $d = ak$ and $a = 5t \Rightarrow d =$
ix $d = ak$ and $a = k \Rightarrow d =$
x $d = ak$ and $a = 2k \Rightarrow d =$

c i $d = \frac{1}{2}k$ and $k = \frac{1}{2}a \Rightarrow d =$
ii $f = \frac{3}{8}g$ and $g = \frac{1}{2}b \Rightarrow f =$
iii $f = 4gh$ and $g = 2a \Rightarrow f =$
iv $g = 4fh$ and $f = 3b \Rightarrow g =$
v $h = 4fg$ and $g = 2f \Rightarrow h =$
vi $f = 4gh$ and $g = \frac{1}{2}h \Rightarrow f =$
vii $f = 4gh$ and $g = \frac{1}{2}h^2 \Rightarrow f =$

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viii $p = \frac{5}{8}ax$ and $a = \frac{1}{4}x \Rightarrow p =$
ix $q = \frac{5}{8}a^2x$ and $a = \frac{1}{2}x \Rightarrow q =$
x $v = \frac{7}{8}ax^2$ and $a = \frac{1}{2} \Rightarrow v =$

Here are some more statements for you to examine. Which ones are true?

$a = 2-4 \Rightarrow a = 4-8$

$b = \frac{3}{7} \Rightarrow b = \frac{21}{49}$

$c = \frac{1}{2} \Rightarrow 2c = 1$

$d = \frac{2}{3} \Rightarrow 3d = 2$

$e = \dfrac{a}{2} \Rightarrow 2e = a$

$f = \dfrac{c}{4} \Rightarrow 4f = c$

$3g = \dfrac{f}{5} \Rightarrow 15g = f$

$6h = \dfrac{p}{7} \Rightarrow 48h = p$

$3h = \dfrac{3p}{7} \Rightarrow 7h = p$

$4k = \dfrac{3}{a} \Rightarrow 4ka = 3$

$a = \dfrac{b}{c} \Rightarrow ac = b$

Exercise 2 Now complete the following statements.

a i $f = \frac{2}{3} \Rightarrow 3f =$
ii $4g = \frac{3}{5} \Rightarrow \qquad = 3$
iii $2x = \frac{5}{7} \Rightarrow \qquad = 5$
iv $3y = \frac{2}{5} \Rightarrow 15y =$
v $4y = \frac{4}{7} \Rightarrow y =$
vi $4y = \frac{4}{7} \Rightarrow \qquad = 1$
vii $4y = \qquad \Rightarrow 28y = 4$

b i $2g = \qquad \Rightarrow 10g = 3$
ii $5h = \frac{2}{3} \Rightarrow \qquad = 2$
iii $7j = \frac{4}{5} \Rightarrow 35j =$
iv $12z = \frac{3}{7} \Rightarrow \qquad = 3$
v $10k = \frac{2}{3} \Rightarrow \qquad = 1$
vi $8u = \frac{1}{4} \Rightarrow \qquad = 1$
vii $6m = \qquad \Rightarrow 72m = 11$

Now examine the following true statements.

$\frac{1}{2}a = 4 \Rightarrow a = 8$
$\frac{1}{2}a = \frac{2}{3} \Rightarrow a = \frac{4}{3}$
$\frac{1}{3}x = \frac{2}{5} \Rightarrow x = \frac{6}{5}$
$\frac{2}{5}y = \frac{5}{3} \Rightarrow y = \frac{25}{6}$

Exercise 3 Complete the following:

a i $\frac{1}{2}b = \frac{3}{7} \Rightarrow b =$
ii $\frac{3}{5}f = \frac{7}{3} \Rightarrow f =$
iii $\frac{3}{2}c = \frac{2}{3} \Rightarrow c =$
iv $\frac{5}{3}v = \frac{3}{2} \Rightarrow v =$
v $\frac{4}{3}m = \frac{4}{5} \Rightarrow m =$
vi $\frac{3}{4}p = \frac{5}{4} \Rightarrow p =$
vii $\frac{1}{3}x = \dfrac{2}{a} \Rightarrow x =$
viii $\frac{1}{2}y = \dfrac{3}{b} \Rightarrow y =$
ix $\frac{1}{5}z = \dfrac{4}{c} \Rightarrow z =$
x $\frac{2}{3}y = \dfrac{5}{d} \Rightarrow y =$

b i $\frac{5}{3}x = \frac{a}{2} \Rightarrow x =$

ii $\frac{4}{3}z = 3a \Rightarrow z =$

iii $\frac{4}{5}p = 2b \Rightarrow p =$

iv $2b = \frac{3c}{d} \Rightarrow b =$

v $\frac{3}{2}q = \frac{2e}{f} \Rightarrow q =$

vi $\frac{3}{5}x = \frac{g}{2h} \Rightarrow x =$

vii $\frac{3}{7}w = \frac{3j}{k} \Rightarrow w =$

viii $\frac{4}{3}z = \frac{t}{4m} \Rightarrow z =$

ix $\frac{3}{2}y = \frac{3n}{p} \Rightarrow y =$

x $\frac{3}{2}y = \frac{3n}{p} \Rightarrow n =$

Reinforcement Exercise 4

Simplify the following:

i $\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$

ii $\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

iii $2x(y+x)-3y(x+2y)-x^2$

iv $3xy(x+y)-2x(x^2-y)+3xy(1+2x)$

v $x^6 \times x^4 \div x^5$

vi $(3y^4+2y^4-y^4)\div y^2$

vii $\sqrt{(x^2+2xy+y^2)}$

viii $\sqrt{8} \times \sqrt{32}$

ix $(13 \times 10^4) \times (2{\cdot}1 \times 10^3)$

x $2001^2 - 1999^2$

Answers on page 233.

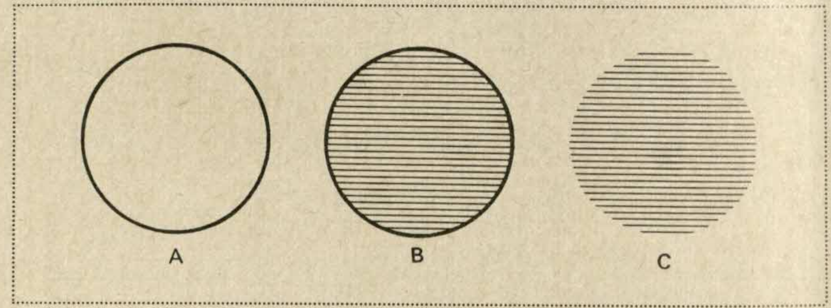
Section 35 **Circles**

Exercise 1 **a** Describe carefully what sort of set a circle is.

b What do you know about the distance from the centre to any point of the line?

c What is the name of the instrument which you usually use for drawing circles?

d Why does that instrument work?



The drawings above represent different things.

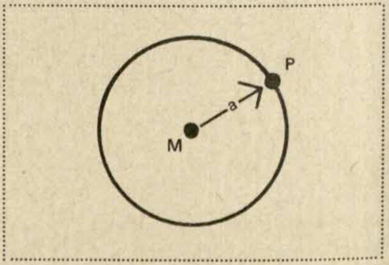
A is a line set. We usually call it *a circle.*

B is the line set together with all the points inside it. We usually call this *a closed disc.*

C is the set of points inside the circle set, the interior of the circle or *an open disc.*

Exercise 2 Suppose the diagrams A, B, C above all have a radius of 3 in. We will call the distance of a point P from a fixed point M the distance *a*.

Each of these descriptions is of either A or B or C above. Say which is which. In each case all the positions of P are in the same plane.



a $\{P|a \leqslant 3\}$ **c** $\{P|a < 3\}$

b $\{P|a = 3\}$ **d** $\{P|a > 3\}$

Exercise 3 Suppose we do not limit the descriptions of Exercise 2 to points in the same plane. What will (a), (b) and (c) describe now? The three words you need are sphere, closed ball and open ball. Which is which?

Exercise 4 Look back at your three diagrams A, B and C above. Each of these has a *perimeter*, a *radius* and an *area*.

a Draw a sketch of a circle and on it mark the perimeter with a red line.

b Beside the diagram write in red another word for the perimeter of a circle.

c Draw in the diagram a black line to show a radius.

d Draw in a blue line to show a diameter.

e If the radius of a circle is three inches find

i the diameter ii the approximate circumference
iii the approximate ratio of the circumference to the diameter
iv the approximate area
v the approximate ratio of the area to the circumference.

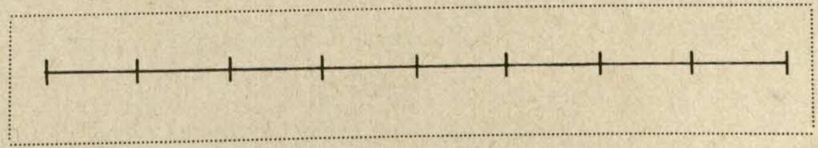
Exercise 5 **a** A boy stands still in the middle of the school field. He is holding one end of a long rope and on the other end of the rope he has fixed a spanner. If he swings the rope round him say what path is traced out by the spanner.

i if he turns so that he is always facing the spanner
ii if he stands still and lets the rope wind round him.

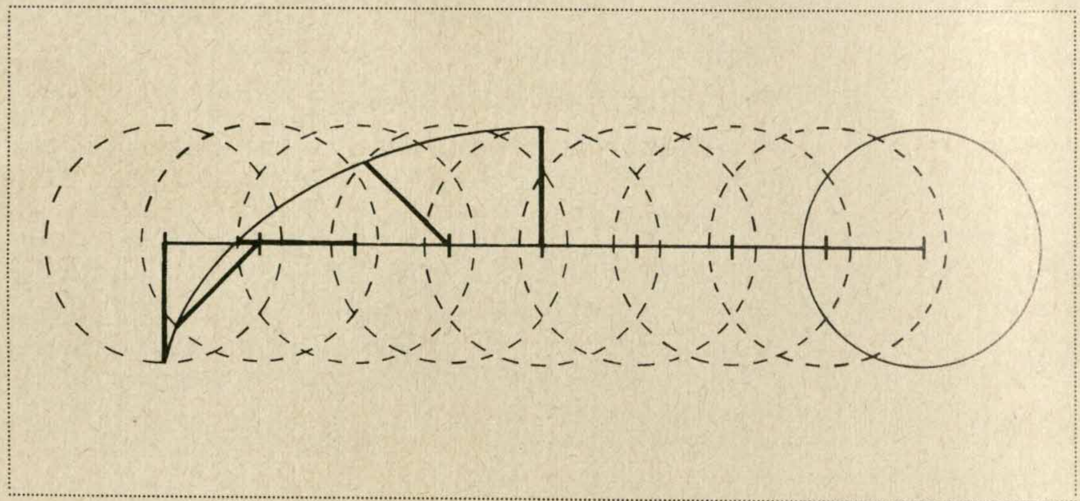
b What word do we sometimes use for the 'path traced out'?

Exercise 6 Here is a drawing for you to do. Draw it very carefully. You will need to be sure that your compasses work properly and your pencils are sharp before you begin.

First draw a straight line segment $12\frac{1}{2}$ in. long leaving at least 3 in. of paper above, below and at the ends of the segment. Divide the segment into eight equal parts.



Now carefully draw circles radius 2 in. at each end of the line at each of the points you have marked along it. Next place radiuses in the diagram as we have done. Each one is 45° beyond the last (looking left to right). The final step is to join up the tips of these radiuses by a smooth curve.



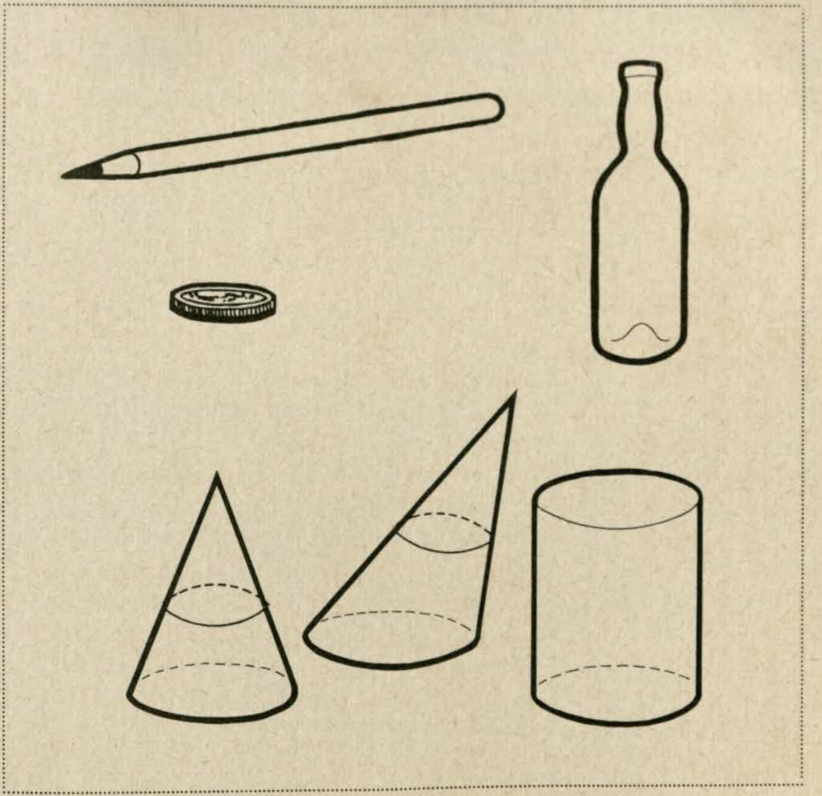
a Suppose your circles represent the positions of a bicycle wheel travelling along a level road. What is the curve you drew last of all?

b Which part of a bicycle wheel is always going backwards when the bicycle is going forwards?

Exercise 7 Repeat the last drawing but join up the points half-way along the radius to make the curve. What do you notice?

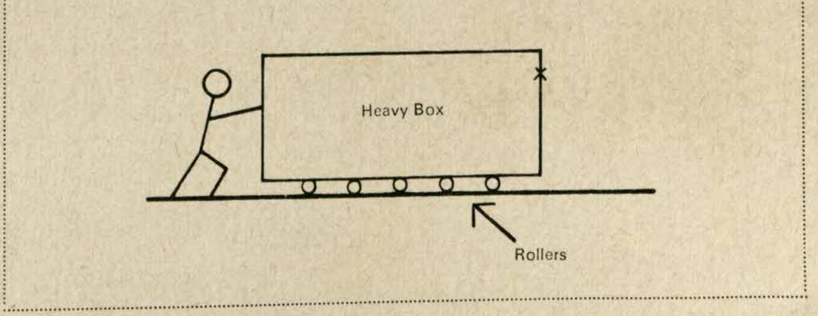
Exercise 8 See if you can now construct the locus of a point outside the circle as the circle rolls along. (You could do this on the same diagram as Exercise 7 if you use a new colour.) What do you notice this time? What is the locus of the centre of the circle in each of the three cases?

Exercise 9 This diagram shows some objects and some geometrical figures which contain circles. Look at them carefully for some of them contain a great many circles. Find them all. Now in the next five minutes make a list of as many common objects as you can which contain circles.

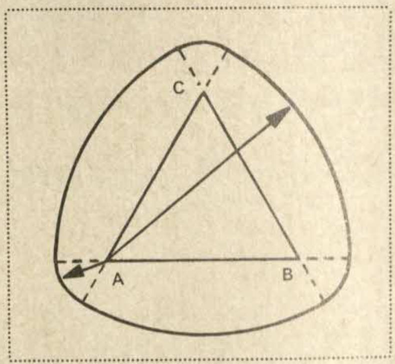


Exercise 10 A circle has one very useful property. Have you ever thought about why rollers for moving heavy objects are round (in section)? Look at the box illustrated. What is the locus of the point marked with a cross as the box is moved?

Why is the locus like that?

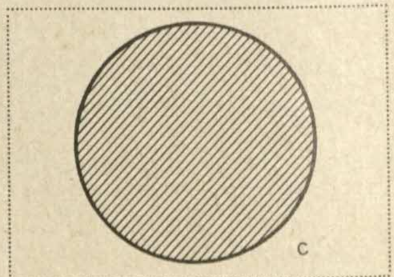


Exercise 11 Is a circle the only figure which has a constant diameter?
Here is a drawing for you to try.

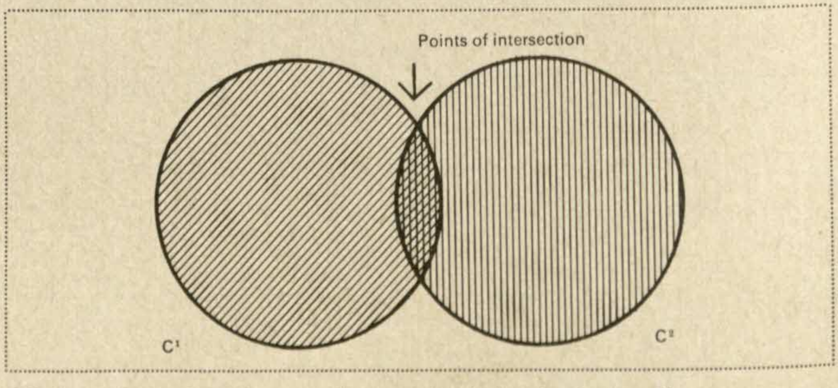


Begin by drawing an equilateral triangle. Produce (extend) the sides, all by the same amount, say half the length again.
Now with centre A draw an arc, and the same arc centres B and C.
Now with centre A draw another arc, and the same arc at B and C.
The figure you have drawn has a 'constant diameter'. What does this mean? If a roller were made with this shape as its section, would it work as well as a circular one? Could you have a wheel made this shape? Would it be as good as a circular one? Explain.

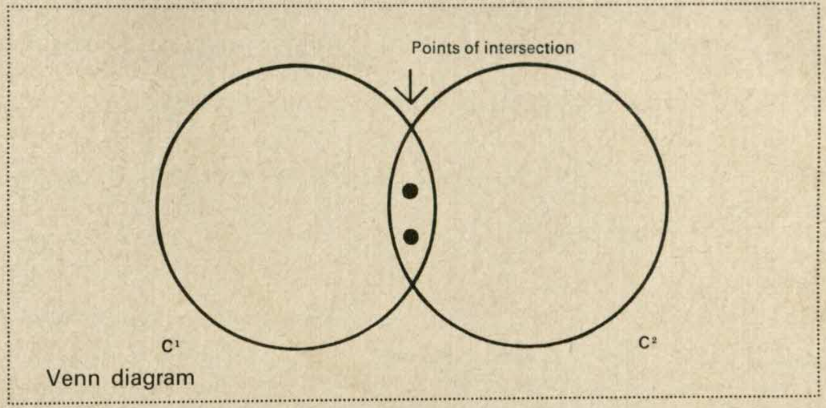
Here is a Venn diagram. It represents a set of points. This particular set of points actually forms a circle set (that is, a line).



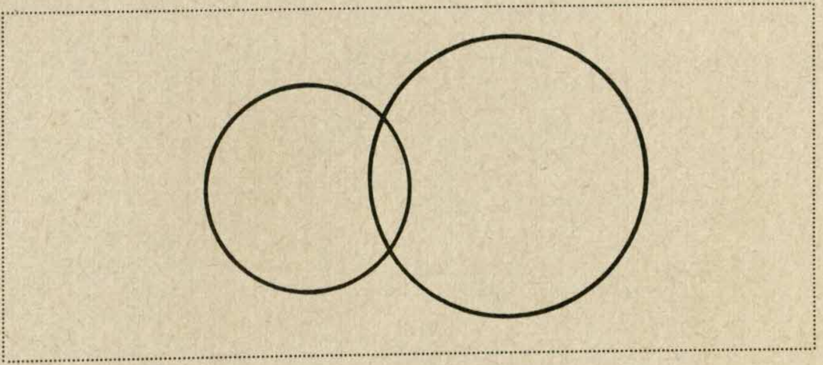
This Venn diagram refers to the intersection of two circles C_1 and C_2.



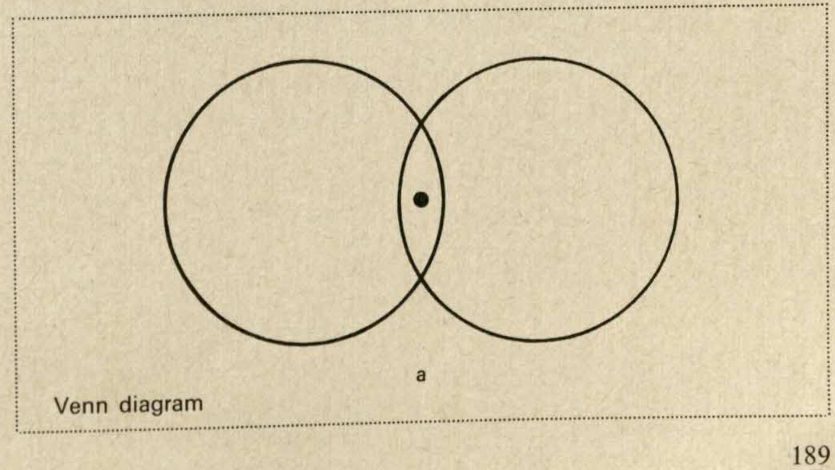
All the points of the intersection are shaded twice in the diagram.

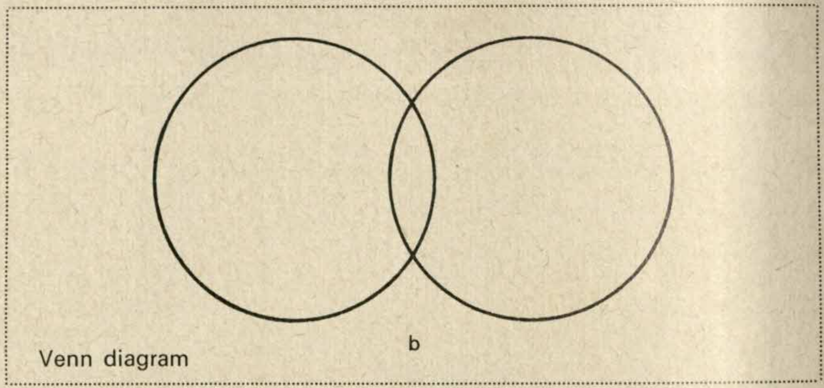


These two circles must look something like this:

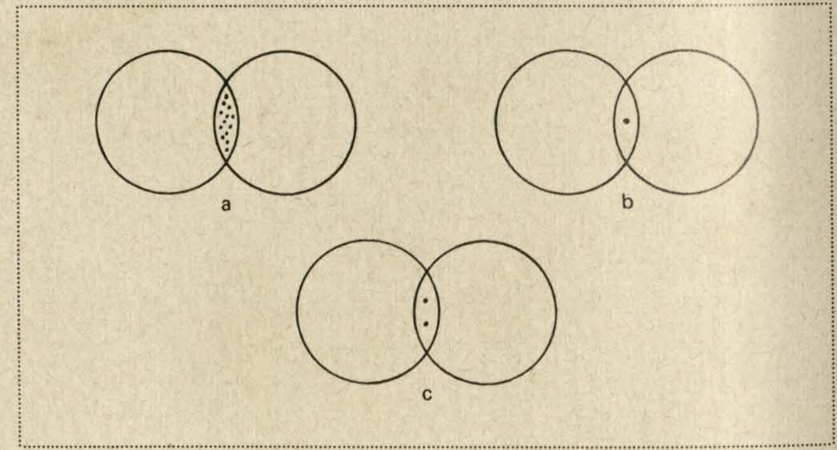


Exercise 12 Draw sketches to show what these circles must look like.



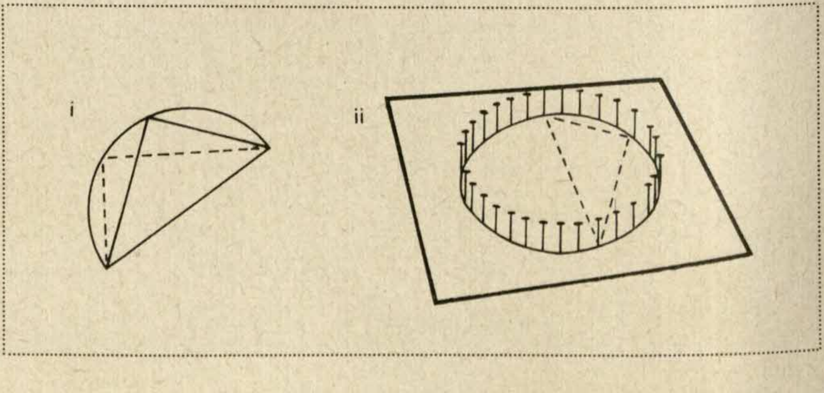


Exercise 13



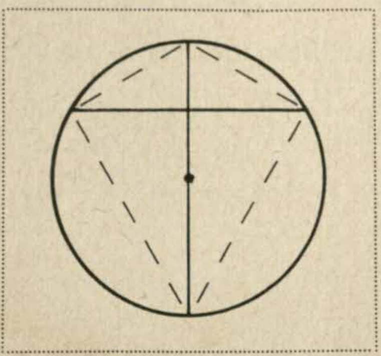
These Venn diagrams show the intersections of two closed discs. Draw pictures of what the discs must look like.

Exercise 14 a

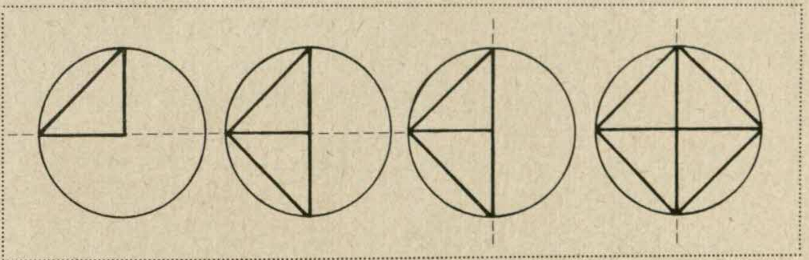


The diagram (i) shows a half circle or semi-circle, together with some chords. What important fact can you discover about angles in the diagram? You may use a board like the one in the diagram (ii) to find out.

b Have you seen the shape inside the circle in this diagram before? What can you say about its symmetry? What other facts do you know about it?

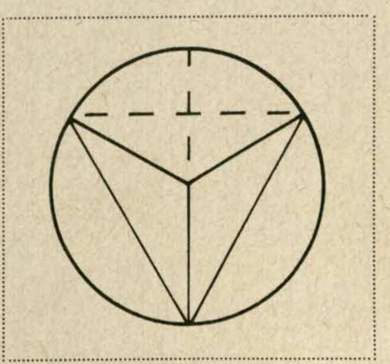


c Look at the following strip cartoon and then answer the questions below.



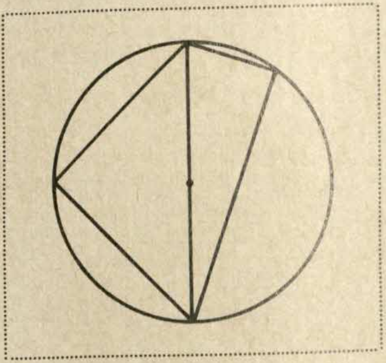
i What sort of figure is contained inside the circle in the last diagram?
ii What facts do you know about the figure?
iii What can you say about symmetries this time?
iv What can you say about the four small triangles in the figure?

d Look at the following figure. Have you seen something like it before?

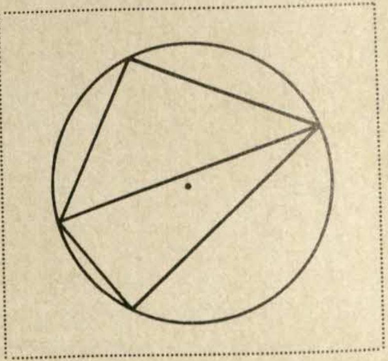


- i Copy the diagram and measure the angles in your figure. What facts do you notice?
- ii What sort of triangle is the big one?
- iii What sorts of triangles are the smaller ones? (Firm lines.)
- iv What can you say about the triangles with dotted lines?
- v What can you say about symmetry in the diagram?

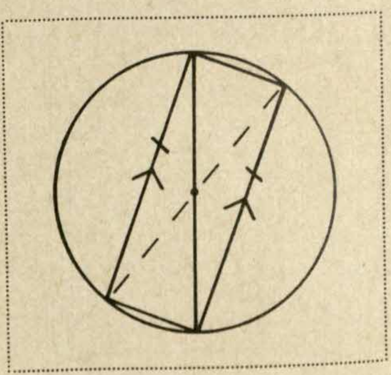
e What can you say about (i) angles and (ii) symmetry in this diagram?



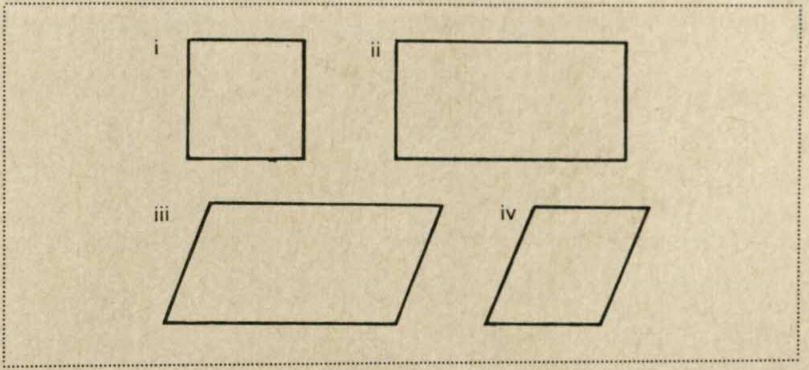
f What can you say about (i) angles and (ii) symmetry in this diagram?



g What can you say about (i) angles, (ii) lines, and (iii) symmetries in this diagram?

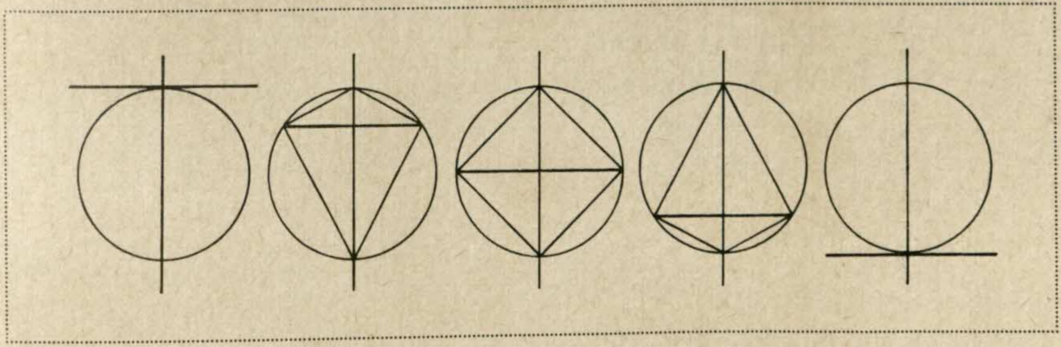


h These diagrams are of (i) a square, (ii) a rectangle, (iii) a parallelogram, and (iv) a rhombus.



Is it possible to draw a circle to pass through all four vertices (corners) in each case? If so, where is the centre of the circle?

i Look at the following strip cartoon. Can you explain what it means? Why are the first and last diagrams different in appearance from the rest? Are the first and last really right?



Section 36 **Estimation: Martian arithmetic**

Here are some statements written down by a Martian space traveller. Read them and decide what he means by the square brackets.

$[2\frac{1}{4}] = 2$; $[4\frac{1}{2}] = 4$; $[3{\cdot}01] = 3$; $[2\frac{5}{3}] = 3$; $[\frac{3}{4}] = 0$.

Exercise 1 **a** Write down the answers to the following:

i $[1\frac{3}{4}] =$
ii $[1\frac{1}{4}+3] =$
iii $[\frac{12}{5}] =$
iv $[\frac{1}{2}+2\frac{1}{2}] =$
v $[\frac{1}{2}]+[2\frac{1}{2}] =$
vi $\left[\dfrac{12+2}{3}\right] =$
vii $\left[\dfrac{5\frac{2}{3}}{4}\right] =$
viii $[1{\cdot}242+2{\cdot}321] =$
ix $[\sqrt{10}] =$
x $[\frac{59}{10}] =$

b If a and b are any numbers, is $[a]+[b]$ always equal to $[a+b]$? Is it necessarily different?

c Write down the answers to the following using as little working as possible.

i $[\frac{1296}{1295}] =$
ii $[\frac{49}{50}] =$
iii $[\frac{1}{2}+\frac{1}{3}] =$
iv $[\frac{1}{2}+\frac{2}{3}] =$
v $[2\frac{4}{7}+1\frac{1}{2}] =$
vi $[4\frac{3}{8}+2\frac{3}{4}] =$

d i $\dfrac{5\frac{1}{2}}{11} =$?
ii Is $\dfrac{5}{11} > \dfrac{1}{2}$?

e Look carefully at the position of the brackets in each of the following. Simplify them by the shortest method you can.

i $[\frac{23}{5}+\frac{25}{4}+\frac{15}{4}]$
ii $[\frac{23}{5}]+[\frac{25}{4}+\frac{15}{4}]$
iii $[\frac{1}{2}\times 2\frac{1}{2}]$
iv $[1\frac{1}{2}]\times[2\frac{1}{2}]$
v $[1\frac{1}{2} \div \frac{1}{2}]$
vi $[2\frac{1}{4} \div 1\frac{1}{2}]$
vii $[2\frac{1}{4}] \div [1\frac{1}{2}]$

f Why has $[1\frac{1}{2}] \div [\frac{1}{2}]$ no answer?

g Can you express in words the meaning of the square brackets?

Exercise 2 In the following exercise the two signs $\square$ and $\bigcirc$ will be used for numbers. The problem is to replace each by suitable numbers to satisfy the condition stated. Whatever you use instead of $\square$ on the left must also be used on the right; similarly with $\bigcirc$.

a $[\square]+[\bigcirc] < [\square+\bigcirc]$ **c** $[\square]+[\bigcirc] > [\square+\bigcirc]$

b $[\square]+[\bigcirc] = [\square+\bigcirc]$

Exercise 3 James has 12 shillings and John has 8 shillings. They have become fond of a weekly series of ten booklets which cost 2s. 3d. each. If they pool their funds, can they afford the whole series between them? How many can they get?

Exercise 4 The brackets act as a digit selector. Complete the following:

a $[231] =$

b $[\frac{231}{10}] =$

c $[\frac{231}{100}] =$

d $231-10\times[\frac{231}{10}] =$

e $[\frac{231}{10}]-10\times[\frac{231}{100}] =$

Exercise 5 Comment about the values of $\{x|2 < x < 3\frac{1}{2};\ x \text{ is a whole number}\}$ and $[3\frac{1}{2}]$.

The Martian arithmetic also made some other statements. Read them carefully and decide what they mean.

$(3\frac{1}{2}) = 4;\quad (5\cdot 2) = 6;\quad (7) = 7;\quad (\frac{2}{3}) = 1.$

Exercise 6 **a** Write down the values of

i $(4\frac{1}{4})$

ii $(5\frac{7}{8})$

iii $(2{\cdot}1)$

iv $(3{\cdot}5)$

v $(2\frac{3}{8})-[2\frac{3}{8}]$

b Which is nearer to $2\frac{1}{4}$, $[2\frac{1}{4}]$ or $(2\frac{1}{4})$?
to $3{\cdot}6$, $[3{\cdot}6]$ or $(3{\cdot}6)$?
to $4{\cdot}51$, $[4{\cdot}51]$ or $(4{\cdot}51)$?
to $6{\cdot}5$, $[6{\cdot}5]$ or $(6{\cdot}5)$?

Back to Earth again now!

Exercise 7 **a** Rearrange the following symbols to make true statements.

i 3, 4, 3·4, <, <.

ii 5, 6, 5·6, <, <.

iii 2·4, 2·3, <, 2·39, <.

iv 763, 764, 763·6, <, <.

v 2001, 2000, 2000·5, <, <.

b From the following set A, form two subsets B and C so that $A = B \cup C$ and $B \cap C$ is the empty set. Numbers nearer to 2 than 3 must be assigned to set B and numbers nearer to 3 must be assigned to set C.
$A = \{2, 2{\cdot}2, 2{\cdot}4, 2{\cdot}6, 2{\cdot}8, 3\}$.

c How many decimal numbers can be found between 2 and 3? If we take all these numbers as set A and form sets B and C as in the previous question, how many numbers will be in sets B and C?

d If we form the sets of (c) above, into which sets shall we have to place the following numbers? 2·15, 2·97, 2·4, 2·54, 2·801, 2·4999, 2·5000001. Is it possible to decide into which set to place 2·5?

e Consider the set of numbers between 7·83 and 7·84.
Form two subsets as before. Write in three of the numbers in each set, putting in dots to indicate the rest.
Say whether the following numbers fall in set B or set C. 7·836, 7·832, 7·839, 7·833, 7·8351, 7·8345, 7·831999.

f Divide the set of numbers between 4500 and 4600 into two sets consisting respectively of those numbers nearer to 4500 and those nearer to 4600. For each of the following numbers say which are nearer to 4500 and which are nearer to 4600, 4567, 4598, 4559, 4537, 4583, 4550, 4590, 4529, 4538.

You have learned how to round off numbers. If you want to round off 2350 to the nearest thousand the result is 2000. If 65·73 is given to the nearest whole number, it is 66. If 17·46 is given to the nearest tenth, it is 17·5; if the same number is given to the nearest whole number, it is 17.

Exercise 8 Consider the numbers 24·37, 17·49, 15·01, 107·45, 459·29.

a Give each to the nearest whole number.

b Give each to the nearest ten.

c Give each to the nearest tenth.

d Give each to the nearest hundred.

As you saw in Exercise 8(d) above, you can decide which answer to give to 107·45 to the nearest tenth. There is no answer to the question in fact. There are times when we need to give one for all that. We therefore often agree to regard a 5 as belonging to the same set as a 6. Thus 107·45 to the nearest tenth is given as 107·5.
The processes you did in Exercise 8 are sometimes described as 'correcting to the nearest... ' I think this is a curious description? Why?
237·4539 can be given as:
237·454 to the nearest thousandth or correct to 3 decimal places

237·45	to the nearest hundredth or correct to 2 decimal places
237·5	to the nearest tenth or correct to 1 decimal place
237	to the nearest whole number
240	to the nearest ten
200	to the nearest hundred.

Exercise 9 Give each of the following numbers as instructed.

a 648·43
- i nearest hundred
- ii correct to one decimal place

b 75·245
- i correct to two decimal places
- ii to nearest tenth
- iii to nearest ten
- iv to nearest hundred.

c 20·051
- i to nearest hundredth
- ii to nearest whole number
- iii correct to one decimal place
- iv to nearest hundred.

d 1·0000245
- i correct to 5 places of decimals
- ii correct to 4 places of decimals

Exercise 10 **a** The postal charges for inland letters are as follows: the first 2 oz. costs 4d.; each additional 2 oz. up to 1 lb. costs a further 2d.; thereafter each additional 2 oz. costs 3d. This means that a letter weighing 6 oz. will cost $(4 + 2 \times 2)$ pence, and one weighing $6\frac{1}{2}$ oz. will cost $(4 + 3 \times 2)$ pence. Work out the cost of the following letters:

i $5\frac{1}{2}$ oz. ii $12\frac{3}{4}$ oz. iii 1 lb 5 oz. iv 1 lb $10\frac{1}{2}$ oz.

b The postal charges for letters to the British Commonwealth are: 4d. for the first ounce and $1\frac{1}{2}$d. for each additional ounce or part of an ounce. Work out the costs of the letters of the last question if they had been addressed to Canada.

c The charges for letters sent by sea to all foreign countries are: the first ounce 6d. and each additional ounce 4d. Work out the charges for the same letters to be sent to the U.S.A.

Exercise 11 The Post Office Savings Bank pays interest in the following way: $\frac{1}{2}$d. interest is paid for each complete £ left in the bank for a complete calendar month. This means that £4 12s. 6d. in the bank for 5 months and 26 days earns $(4 \times 5 \times \frac{1}{2})$d. interest. Find the interest on the following deposits.

a £12 9s. 6d. for 13 months and 10 days.

b £54 19s. 0d. for 18 months and two weeks.

c £10 0s. 0d. for 27 days.

d £51 13s. 8d. from 24 January to 8 March in the same year.

e £200 17s. 0d. from 30 August to 30 September in the same year.

f £200 17s. 0d. from 30 August to 1 October in the same year.

g £1 from 31 December 1965 to 1 January 1967.

h £25 deposited on 21 September 1960 and withdrawn on 10 March 1963.

i £*n* deposited for 7 months.

j £*n* left in the bank for *p* complete months.

Reinforcement Exercise 5

Simplify the following:

i $^{+}4+{}^{+}3$
ii $^{-}4+{}^{+}3$
iii $^{-}4-{}^{-}3$
iv $^{+}4+{}^{-}3$
v $^{-}4+{}^{-}3$
vi $^{+}3+{}^{-}4$
vii $^{-}3-{}^{-}4$
viii $^{+}4-{}^{+}3$
ix $^{+}3-{}^{-}4$
x $^{+}3-{}^{+}4$
xi $^{+}3+{}^{+}4$
xii $^{-}3+{}^{-}4$
xiii $^{+}4-{}^{-}3$
xiv $^{-}3+{}^{+}4$
xv $^{-}3-{}^{+}4$
xvi $^{-}4-{}^{+}3$

Answers on page 233.

Section 37 **Right-angled triangles**

Exercise 1 **a** What have these sets of numbers in common?

$\{3, 4, 5\}$ $\{5, 12, 13\}$ $\{7, 24, 25\}$ $\{8, 15, 17\}$

b Each of these sets can be called a *pythagorean triad.* What have these sets to do with geometry?

c Which of the following are pythagorean triads?

$\{5, 6, 7\}$ $\{6, 8, 10\}$ $\{49, 168, 175\}$ $\{40, 75, 85\}$
$\{9, 16, 25\}$ $\{1, 1, \sqrt{2}\}$ $\{1, \sqrt{3}, 2\}$
$\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ $\{\frac{3}{7}, \frac{4}{7}, \frac{5}{7}\}$ $\{\frac{7}{12}, 2, 2\frac{1}{12}\}$

Exercise 2 **a** Find the diagonal of

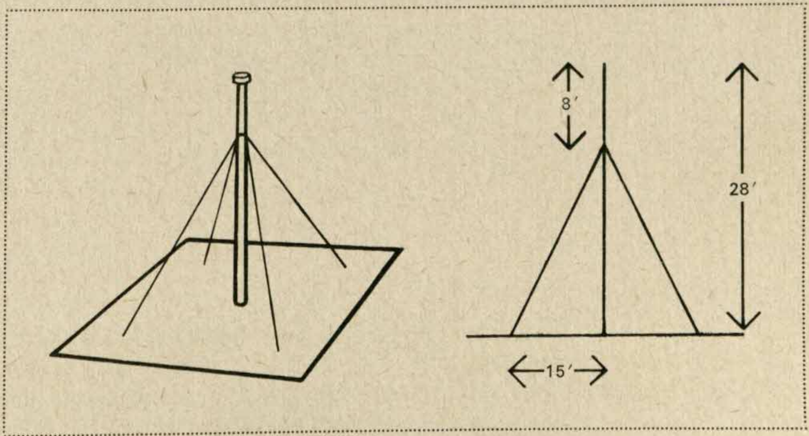
i a rectangle 12 cm. long and 5 cm. wide
ii a rectangle 14 in. by 48 in.
iii a square of side 6 cm.

b If a man travels 16 miles west, and then 30 miles north, how far will he be from his starting point?

c If the man had travelled 1600 miles west from Rome and then 3000 miles north from there, would we still be able to find his distance from Rome by the same method as in (b)? Explain.

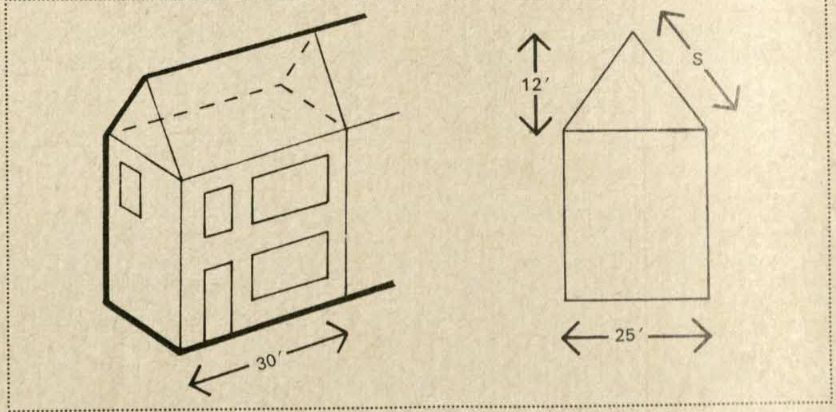
d If the question (b) above were real, could the answer be exact? Explain what your answer means.

Exercise 3 **a**



The diagram shows a flag-staff which stands on a square of turf in a school's grounds. The left-hand diagram shows it in perspective. On the right we have a diagram drawn by taking a 'section' across the diagonal of the square of turf. What length of wire is needed to stay the flag-staff?

b

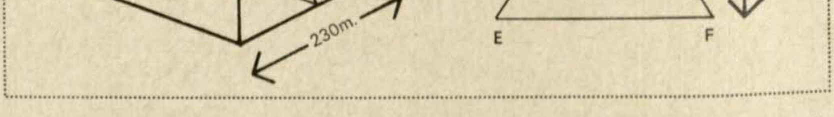


A house is 25 ft. from the front walls to the back. The ridge of the roof is half-way between the front and the back.

The distance from the rafters to the ridge is 12 ft. How long is the slope *s* of the roof? The width of the house is 30 ft. and it is semi-detached. Find the total area of the tiled roof covering the two houses.

Exercise 4

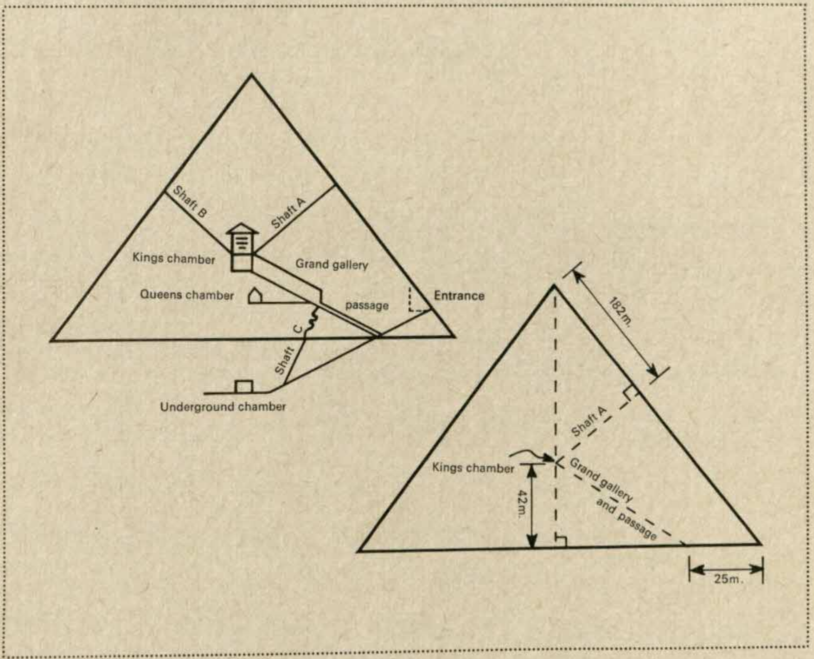
On a rocky plateau in the desert a few miles SW. of Cairo stand three pyramids built by the 4th Dynasty Pharaohs. The biggest is that of Khufu, sometimes called Cheops, and it is probably the biggest single building ever erected. Here are some diagrams of it.



The base is almost perfectly square and the sides face due north, south, east and west. Unfortunately some of the measurements in a reference book are given in English units and some in the metric system. To convert feet to metres, multiply by 0·305.

a i Find the base area in square metres.
ii Find the length of a diagonal of the base in metres.
iii Find the distance from the ground level to the top along the sloping face.
iv Find the length of one of the sloping edges.

b

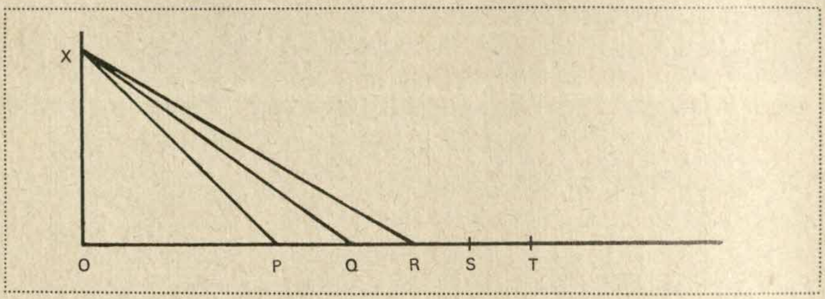


Here is a diagram of the inside of the pyramid. The entrance to the King's Chamber is vertically below the vertex of the pyramid, and is about 42 m. from the ground level. The combined length of the Grand Gallery and sloping passage is not given, but the passage starts at about ground level, 25 m. from the outside of the base. The second diagram shows the first one simplified and the distances are simplified for you. Now work out these distances each to the nearest metre.

i the length of the Grand Gallery and its sloping passage together,
ii the distance of the King's Chamber below the vertex,
iii the length of shaft A.

c About 2,300,000 blocks of stone of average weight $2\frac{1}{2}$ tons went into this structure. How heavy is the entire building in tons?

Exercise 5 Here is a drawing which I have made. OP = OX = 1 in., and OQ = PX, and so on.

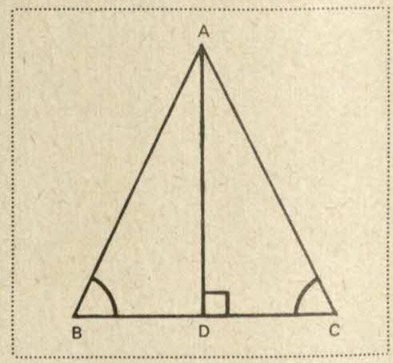


a Use your knowledge of right-angled triangles to complete this table:

OP = OX = 1 in. = $\sqrt{1}$ in.
OQ = PX = in. = $\sqrt{2}$ in.
OR = QX = in. = in.
OS =
OT = etc.

b Now construct this diagram yourself very carefully and carry on until you can read off from it $\sqrt{7}$.

Exercise 6 Find DC if



a AB = 5 in. and AD = 4 in.

b AB = 25 in. and AD = 24 in.

c AB = 25 in. and AD = 7 in.

d AB = 24 cm. and AD = 25 cm.

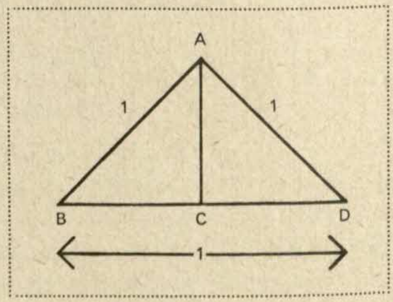
e AB = 2 in. and AD = $\sqrt{3}$ in.

f AB = 8 in. and AD = $(4 \times \sqrt{3})$ in.

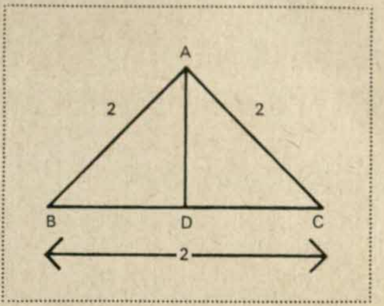
g AB = 4·3 cm. and AD = 1·5 cm.

h Draw the examples (e) and (f). What do you notice?

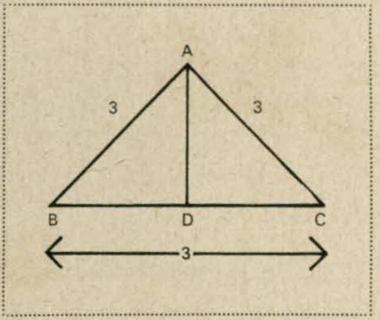
Exercise 7 **a** Find AD.



b Find AD.



c Find AD.



Exercise 8 **a** What is the shortest way of simplifying:

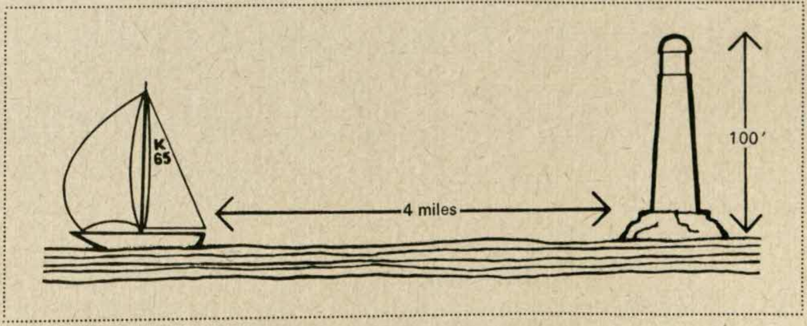
i $985^2 - 5^2$ ii $107^2 - 7^2$?

b Explain the method you used. What is it called?

c Use the same method to simplify

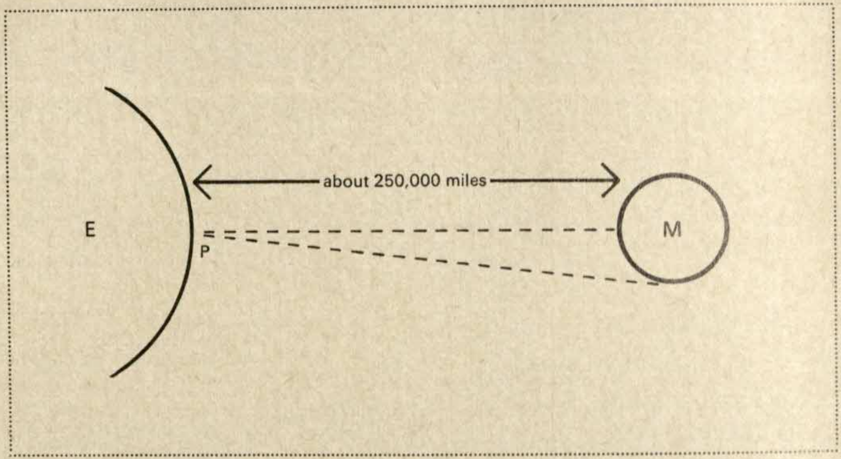
i $0{\cdot}9995^2 - 0{\cdot}4995^2$ ii $25{\cdot}88^2 - 24{\cdot}88^2$

Exercise 9 **a**



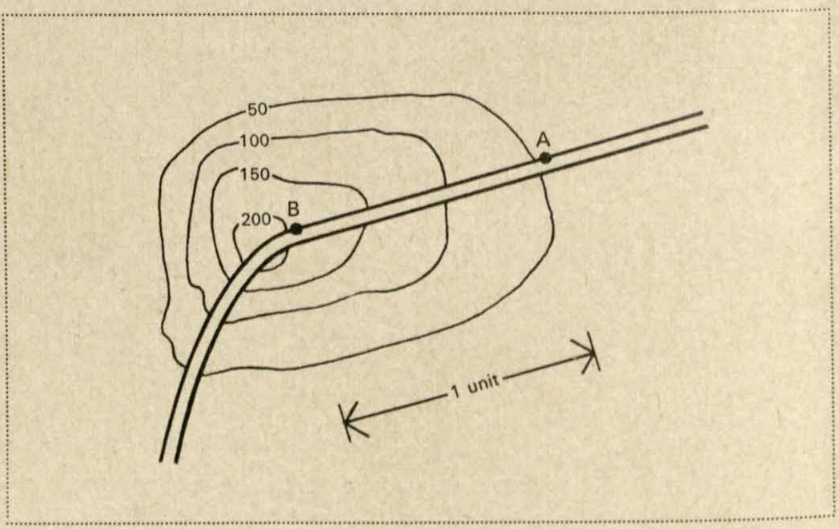
What is the direct distance from the boat to the lighthouse lamp? Is it very different from 4 miles? What do you think this means for practical purposes?

b



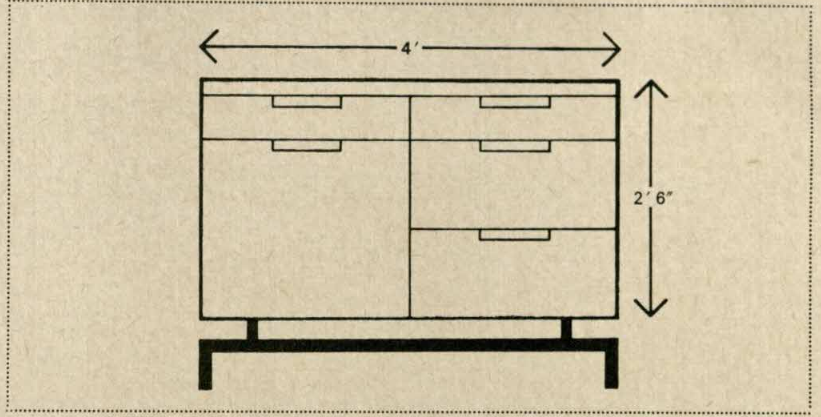
The diagram shows the earth (E) and the moon (M). (Is it drawn to scale?) Do you think it matters whether we measure the distance from P to the moon at the edge or the centre of the moon? Do you think that if you knew these two distances you could use Pythagoras' theorem to find the diameter of the moon accurately?

Exercise 10



Here is part of a map drawn to a scale where 1 unit represents $\frac{1}{2}$ mile. The contours are in feet. If the road between A and B is more or less straight, how much longer is the actual distance AB along the road surface than the distance AB as represented on the map?

Exercise 11



A cabinet maker 'squaring up' the back of a sideboard 4 ft. long and 2 ft. 6 in. high uses a string across the corners. How long will it be if the sideboard is 'true'?

Exercise 12 **a** In rectangle (i) the diagonal is 10 cm. long and one side is 7 cm., and in rectangle (ii) the diagonal is 4·6 in. long and a side is 3·4 in. Find the other sides and the areas of the two rectangles. Which rectangle has the greater area?

b Find the lengths of the sides of squares whose diagonals are

i 2 in. ii 10 cm. iii 14·14 cm.

Can you without further working say what the areas of the squares are?

c Find the lengths of the sides of the rhombuses whose diagonals are

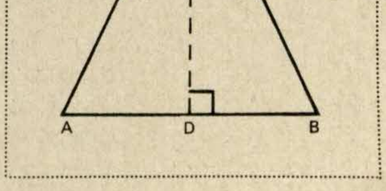
i 16 cm. and 12 cm. ii 4 in. and 6 in.

d Find the heights CD of the isosceles triangles ABC where

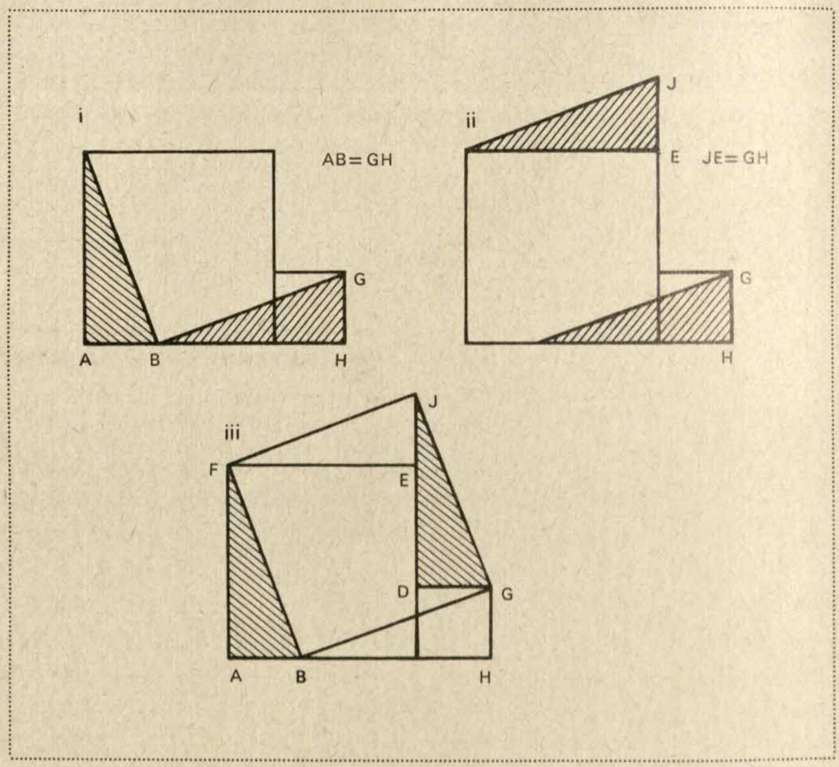
i AB = 4 in., BC = 5 in.

ii AB = 64 mm., BC = 40 mm.

Find also the areas of these triangles.



Exercise 13

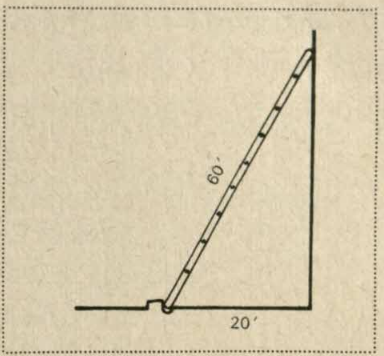


Look at diagram (i) which consists of two squares and two shaded triangles. Explain how you know that the two triangles are equal in area.
Now look at diagram (ii). What can you say about the two triangles in this figure?
Now examine diagram (iii). What can you say about triangle FAB and triangle JDG?
Explain how these facts together can be used to prove Pythagoras' theorem.

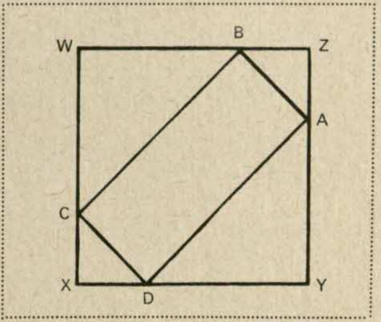
Exercise 14 **a** Two boys are shouting messages to one another across the playground which lies between two buildings. One boy's head is at a window 14 ft. higher than the other's. The playground is 48 ft. wide. How far do the voices actually have to carry?

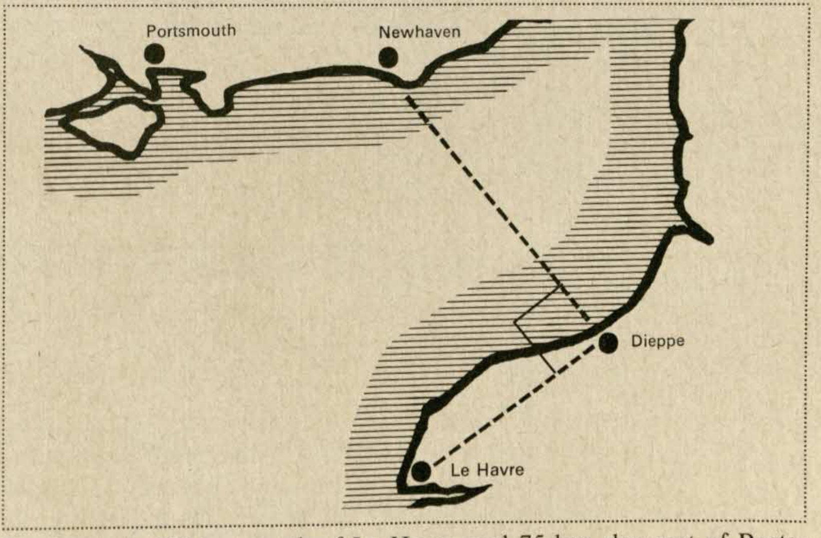
b In a recent TV film, the cops were concealed at the top of a 800-ft. cliff. They were armed with rifles which were aimed at the robbers. The sights were set at 1700 ft. The robbers' old enemies were concealed at the foot of the cliff. They were also armed with rifles. At what range would they have set theirs?

c The caretaker's ladder is 60 ft. long. Since he has no helper to hold the foot of it he has to place it firmly. There is a concrete gutter 20 ft. away from the wall which he can use. How high up the wall will the top of the ladder reach?



Exercise 15 **a** In the diagram WXYZ is a square whose sides are 5 units long. BZ = AZ = CX = DX = 2 units. Show that the figure is symmetrical about an axis. Hence show that ABCD is a rectangle. What is its area?





b Newhaven is 150 km. north of Le Havre and 75 km. due east of Portsmouth. How far is it 'as the crow flies' from Portsmouth to Le Havre? The crossing from Newhaven to Dieppe is about 120 km. How far is it in a straight line from Dieppe to Le Havre?

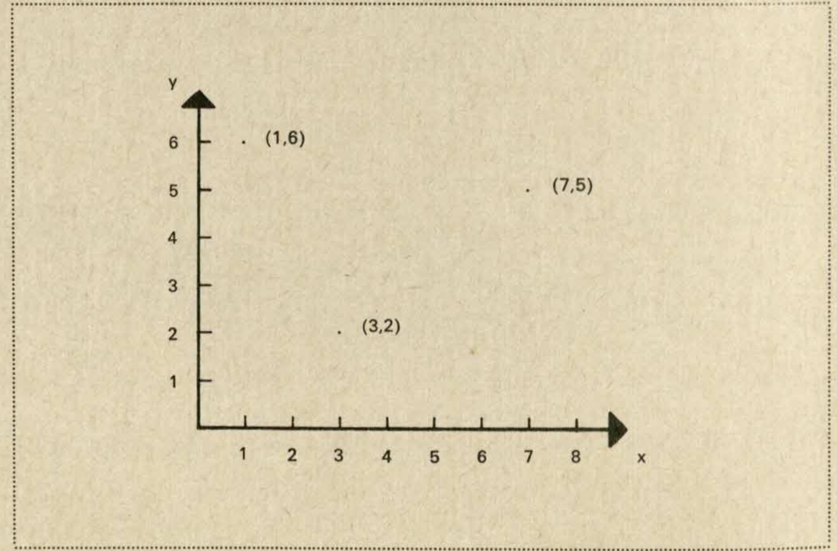
Section 38 **Number pairs (or couples)**

Exercise 1 **a** Write down the members of the following set M putting each pair of numbers into a bracket, the x-number first.
$\{(x, y) | x+y = 5;\ x \text{ and } y \text{ are natural numbers}\}$.

b Write out the first six members of the set
$N = \{(x, y) | x-y = 1;\ x \text{ and } y \text{ are natural numbers}\}$.

c Write out the set $P = \{(x, y) | xy = 20;\ x \text{ and } y \text{ are natural numbers}\}$.

d Write down a description of the following set of number pairs or couples:
$\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.



It is sometimes useful to represent number-pairs as a drawing or graph. This is done like this: two lines at right angles to each other, called *axes*, are drawn. They are each marked off to represent the natural numbers. The 'horizontal' one is used for the x-numbers and is called the 'x-axis'. The 'vertical' one is used for the y-numbers and is called the 'y-axis'. (Remember that the x-number goes into the bracket on the left.) A few number pairs are illustrated as ⊙ in the diagram. Look at them carefully to see that you understand how to find the positions yourself.

Exercise 2 **a** On a piece of graph paper, using any convenient scale, plot the following points:

A(0, 6) B(6, 6) C(6, 0) D(3, 3) O(0, 0).

Calculate how many units apart are O and A, O and B, B and C, O and D, D and C.

b How far apart are the points

i (2, 1) and (1, 3)
ii (0, 0) and (3, 1)
iii (2, 0) and (0, 3)
iv (1·4, 2·3) and (3·3, 1·4)
v $(-1, -1)$ and (3, 1).

Exercise 3 **a** Why have we drawn arrow-heads on the axes? What are lines like this sometimes called?

b How were the positions of the number pairs on the diagram found?

c Where do number pairs like (3, 0), (5, 0), (6, 0), (7, 0) lie?

d What sort of number pairs will be illustrated by points on the *y*-axis?

e Look at the following sets:

$X = \{(0, 0), (1, 0), (2, 0), (3, 0) \ldots\}$
$Y = \{(0, 0), (0, 1), (0, 2), (0, 3) \ldots\}$

What do you think they are?

f Write down the set $X \cap Y$. Which points does this represent on the diagram?

Exercise 4 Draw a pair of axes on $\frac{1}{2}$-in. squared paper. Mark the positions of the two axes, putting them half an inch apart. Mark the positions of the following points on your graph: (3, 2), (4, 4), (4, 2), (2, 2), (2, 4), (3, 3), $(2\frac{1}{2}, 2)$, $(3\frac{1}{2}, 2)$.

a What did you have to do to place the last two?

b What do you notice about a set of points whose *y*-numbers are the same? Do you think a similar thing would be true of a set in which the *x*-numbers were the same?

c What do you notice about the points in which the *x*-number is equal to the *y*-number?

d Do you think the idea of placing the numbers along the axes could be extended to include *all* the fractions? If so, explain how? How could you fit in $(5\frac{34}{35}, 1\frac{10}{17})$ for instance?

The two numbers in a number pair have a special name when they are used in connexion with graphs. They are called respectively the *x*- and

y-coordinate. (They used to be called 'abscissa' and 'ordinate' and these words are still used sometimes.) Placing the points onto a diagram is called 'plotting' the points.

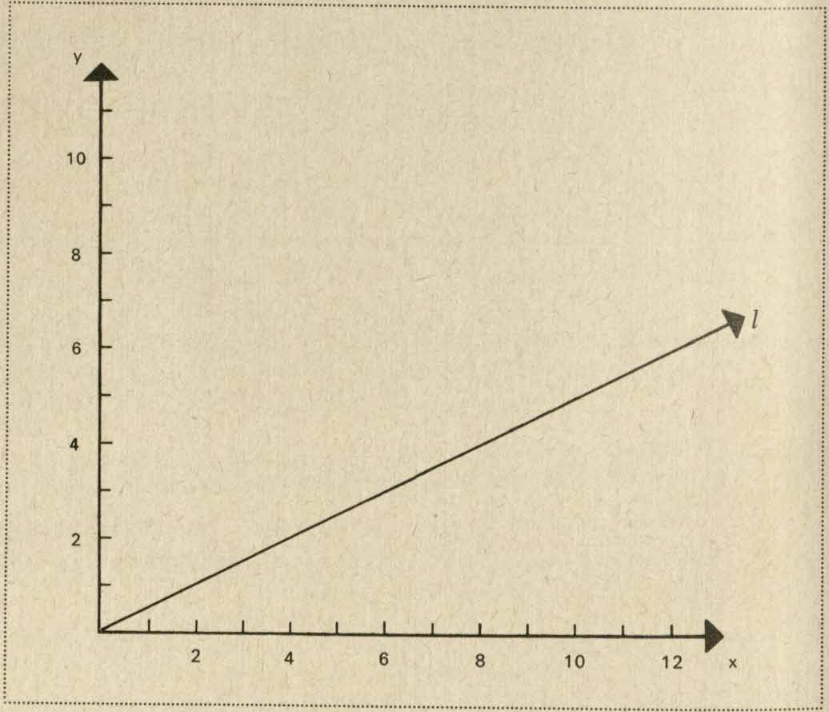
Exercise 5 Look at the following number pairs: (2, 4), (3, 6), $(1\frac{1}{2}, 3)$, $(\frac{1}{4}, \frac{1}{2})$, (1, 2), (4, 8).

a What do you notice about the relationship of the x-coordinate and the y-coordinate in each case?

b Plot these points on a piece of squared paper, using convenient divisions to represent natural numbers on the axes. Draw a *continuous* line through these points. What sort of line can you draw?

c Does the line in your diagram only cover points in the original set? Explain.

d Write down a set description of the points on this line l:



Exercise 6 a $\{(x, y)|x = 2y\}$. (i) Do you think this description is correct for the *points* plotted on the diagram? (ii) Do you think it is a description of l?

b Why has l an arrowhead on it?

c Is the number pair (0, 0) represented on this line?

d Write down the number pairs for the points plotted.

e The following are incomplete number pairs which also lie on the line l:
(5,), (7,), (, $5\frac{1}{2}$), (, 24), (17,).
Fill in the blank spaces.

f Could you plot the last two points on the diagram as it stands above? What would you have to do to plot them?

g Can you identify the set $\{(x, y) | y = \frac{1}{2}x\}$ on this diagram?

The diagram above shows the guide lines forming the squares which you need in order to know where to place the points. An arrangement of threads or strips which form a regular pattern like this is often called a 'lattice' and this is the word we use for this particular purpose. Your graph paper shows a fairly fine lattice. Any point in the plane appears somewhere in the lattice, if you extend it far enough. If we were able to draw a lattice covering every point of the plane, the lines would be very dense. Indeed they would cover the whole plane completely. For our purposes we use only a selected few of the lines and we have to estimate the position of points not lying exactly on the intersections of the lattice.

Exercise 7 **a** Work out a set of six number pairs to fit the description $\{(x, y) | y = 3x\}$. Choose small natural number values for the variable x and then work out y.

b Plot the *whole* set $\{(x, y) | y = 3x\}$ as a line on graph paper.

c Plot the sets $\{(x, y) | y = 2x\}$ and $\{(x, y) | y = \frac{1}{2}x\}$ on the same sheet of paper as part (b). (You could use different colours for each of the lines.)

d What do you notice about the way the lines slope on the paper? Does this have any connexion with the number before the x-term in the equation?

e Do any of the equations contain powers of x or y?

Exercise 8 **a** How many straight lines can be drawn in space passing through one particular point?

b How many straight lines can be drawn in a plane and passing through a single point?

c How many straight lines can be drawn in space but passing through two particular points?

d How many straight lines can be drawn in a plane so that they pass through two particular points?

e If we know that a graph is to be a straight line, how few points can we manage with to fix the position of the line?

f If one of our points were calculated incorrectly and we relied on the minimum number of points, what would happen to our line? How do you think we might guard against this?

g Do you think it matters which points we use?
Explain.

We often shorten the way we say things in mathematics to reduce the amount of writing we have to do. In this topic, we often say 'Plot the equation $y = 2x$' instead of 'Plot the set $\{(x, y)x = 2y\}$ as a graph'. To simplify the writing of number pairs we often tabulate them:

x	1	4	5	etc.
y	2	8	10	etc.

The number pairs here are (1, 2), (4, 8), (5, 10) etc.

Exercise 9 On separate sheets of graph paper, plot the following graphs, making a table of a sufficient number of values for each one.

a $y = 4x$

b $y = \frac{3}{4}x$

c $y = x+1$

d $y = 2x+3$

e $y = 3x+2$

f $y = \frac{1}{2}x+4$.

So far we have only considered positive values of x and y. Is there any reason why we should limit ourselves in this way?

Exercise 10 **a** Write out six members of the set (three using negative integers for x):
$\{(x, y)|y = 2x-4;\ x \text{ is an integer}\}$.

b Draw a graph of the set in (a).

c In a new colour, draw the graph of the set:
$\{(x, y)|y = 2x-4\}$.

Your graphs should look something like the diagram opposite:
You will notice that to include values which are negative, we can extend the axes indefinitely in either direction.

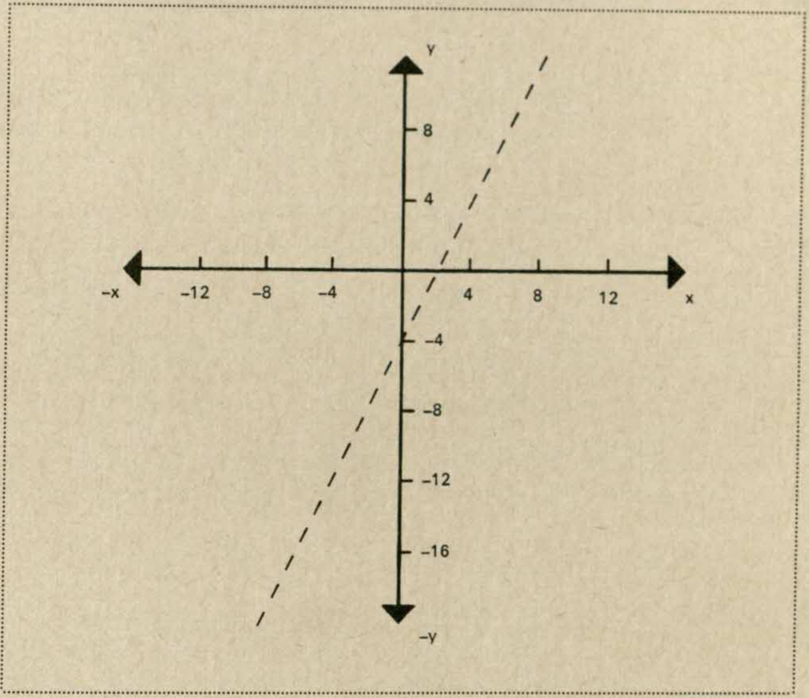
Exercise 11 Draw the following graphs using negative values as stated in each question. (You may use the smallest safe number of values in each.)

a $y = 3x-2$; let x lie between $^{-}5$ and $^{+}6$.

b $y = \frac{1}{2}x-4$; $^{-}4 \leqslant x \leqslant {}^{+}10$.

c $y = \frac{3}{8}x + 1;\ ^{-}16 \leqslant x \leqslant {}^{+}24.$

d $y = \frac{5}{6}x - 4;\ ^{-}24 \leqslant x \leqslant {}^{+}12.$



Section 39 **More about subsets of the plane**

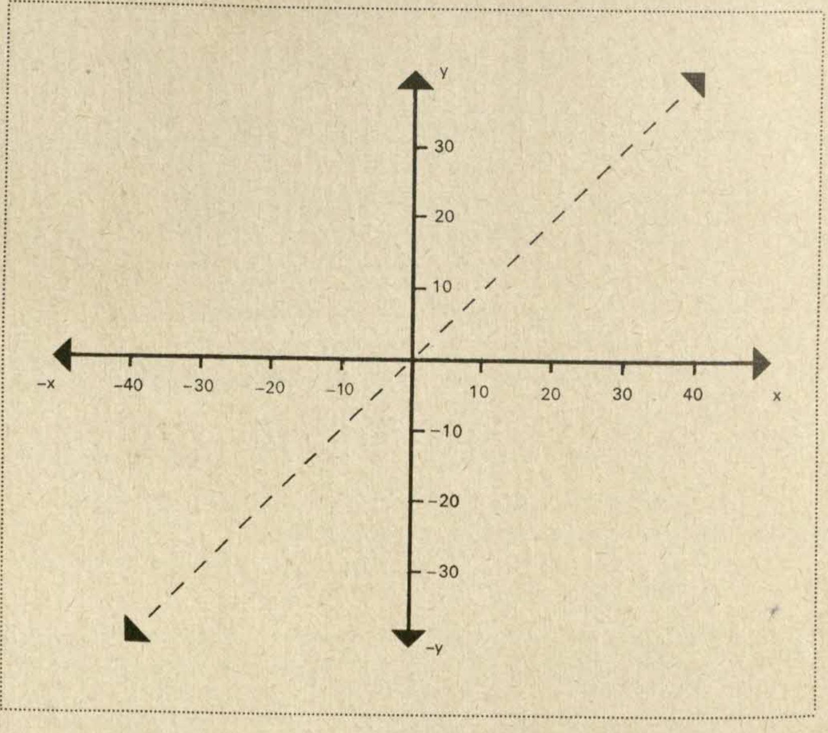
We have already learnt how to represent number pairs as points on a plane. We learnt that there is only one number pair for every point and only one point for every number pair.

Exercise 1 **a** Plot the points $(2, 6)$, $(3\frac{1}{2}, 7\frac{1}{4})$, $(-3, 4\frac{1}{8})$, $(-4\frac{1}{4}, -2\frac{3}{8})$, $(2\frac{5}{8}, -5\frac{5}{8})$ on a piece of graph paper using a suitable scale.

b Plot the point $(2, 3)$ and its reflection in the x-axis $(2, -3)$; also plot the point $(6, -4)$ and its image $(6, 4)$.

c Write down the coordinates of the reflections of the following points: $(4, 2)$, $(-3, 4)$, $(3, -4)$, $(-4, 2)$, $(-3, -4)$.

Exercise 2 The diagram shows the set of points $\{(x, y) | y = x; x \text{ is a real number}\}$



a If x can be any real number, what sort of values can y have?

b Look at the point P on the diagram. Is the y-coordinate equal to the x-coordinate? Is it greater or is it smaller?

c Shade //// the set $\{(x, y)|y > x; x \text{ is real}\}$.

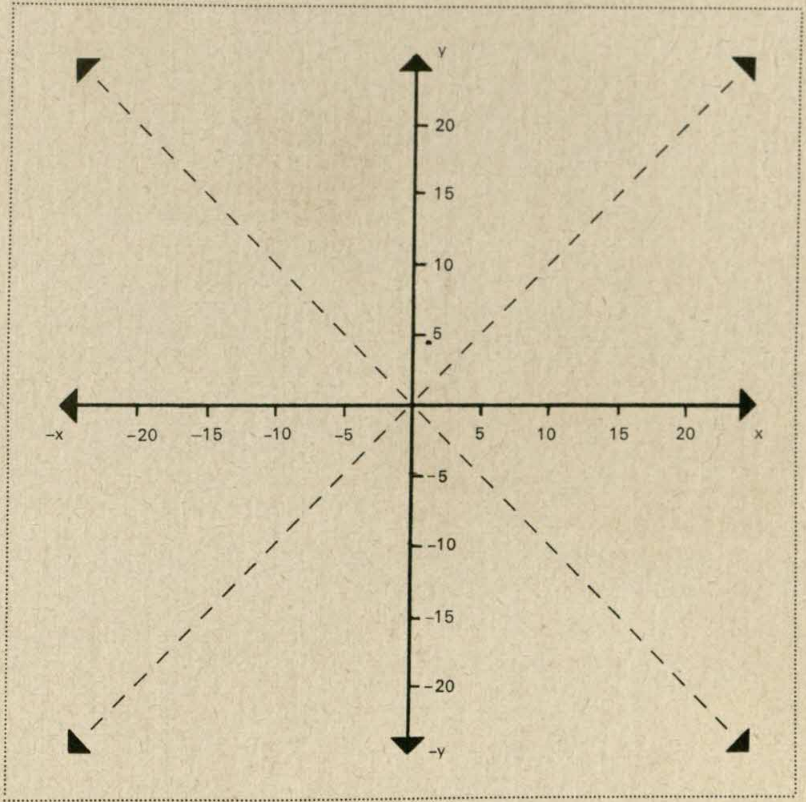
d Shade \\\\ the set $\{(x, y)|y < x; x \text{ is real}\}$.

e Into how many subsets does the line $y = x$ divide the plane?

Exercise 3 The diagram shows the sets of points

$$l_1 = \{(x, y)|y = x; x \text{ is a real number}\}$$
$$l_2 = \{(x, y)|y = -x; x \text{ is a real number}\}.$$

Copy the diagram.



a Put a ring round the set $l_1 \cap l_2$.

b Shade ≡ the set $\{(x, y)|y > x; x \text{ is real}\}$.

c Shade |||| the set $\{(x, y)|y < -x; x \text{ is real}\}$.

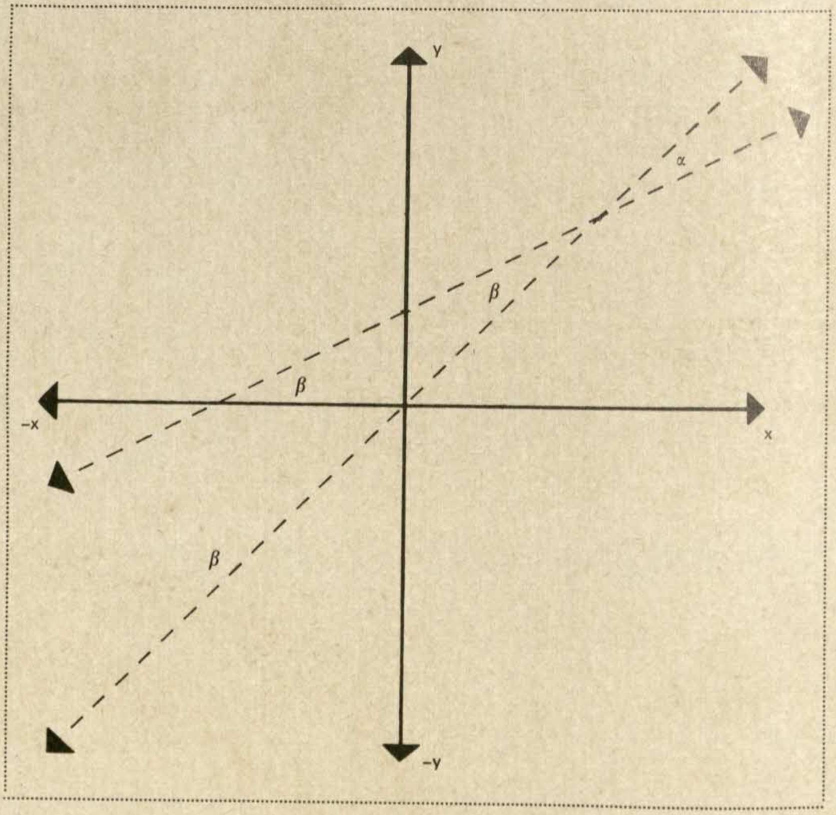
d Identify the set of points which are $\{x, y \mid y > x; y < -x; x \text{ is real}\}$.

e Identify each of the following sets by shading them in different colours:
$\{(x, y) \| y < x; y > -x; x \text{ is real}\}$
$\{(x, y) \| y > x; y > -x; x \text{ is real}\}$
$\{(x, y) \| y < x; y < -x; x \text{ is real}\}$.

f Which is the set $\{(x, y) \| y = 0; x \text{ is real}\}$?

g How is your set $\{(x, y) \| y < x; y > 0; x \text{ is real}\}$ shaded?

Exercise 4 The diagram shows the set $l_3 = \{(x, y) \| y = x; x \text{ is real}\}$ and $l_4 = \{(x, y) \| y = \frac{x}{2} + 1; x \text{ is real}\}$.



a Put a ring round $l_3 \cap l_4$.

b Write down the coordinates of the intersection set $l_3 \cap l_4$.

c Shade ≋ the set $\{(x, y) \| y > x; y\frac{x}{2} + 1; x \text{ is real}\}$.

d Give set descriptions of the regions marked α and β in the diagram.

Exercise 5 **a** Draw fairly accurate diagrams on squared paper to illustrate the following sets of ordered pairs (x, y) such that:

i $y = 2x$
ii $y > 3x$
iii $y \geqslant \frac{1}{3}x$
iv $y < \frac{1}{4}x$
v $y = -2x$
vi $y > -3x$
vii $y < x + 3$
viii $y > 2x - 3$
ix $y = 4x + 6$

x is real in each case.

b Draw graphs on graph paper to determine

i $A \cap B$ where $A = \{(x, y) | x = 3y\}$
and $B = \{(x, y) | y = x - 2\}$
ii $C \cap D$ where $C = \{(x, y) | y = 3x\}$
and $D = \{(x, y) | y = 4x - 1\}$
iii $E \cap F$ where $E = \{(x, y) | y = x\}$
and $F = \{(x, y) | y = 3x - 4\}$
iv $G \cap H$ where $G = \left\{(x, y) \middle| y = \dfrac{3x}{2}\right\}$
and $H = \{(x, y) | y = 2x - 2\}$

x being real in each example.

c Draw sketches to show the following regions. In each of the sets x is real.

i $y > 3x$ and $y < x - 2$.
ii $y < 5x + 1$ and $y > \frac{1}{2}x - 2$.

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Section 40 **Index sets**

$$\begin{array}{rccccl} A = \{ & 2, & 4, & 6, & 8, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ W = \{ & 1, & 2, & 3, & 4, & \ldots\} \end{array}$$

Exercise 1 **a** Can you carry this mapping on for as long as you like?

b What element in A will correspond to each of the following elements of W?

i 7 ii 12 iii 47 iv 1257

c What element of W will correspond to each of the following elements of A?

i 14
ii 36
iii 68
iv 246
v 5972

d Is there always an element of W to correspond with any element you care to choose in A?

e Does every element of A have an image in W?

f When two sets can be mapped into each other in this way we say they have the same... (supply the missing word.)

Your last piece of reasoning tells you that there is the same number of even numbers as natural numbers. Do you think this is really rather surprising?

W forms a useful *index* to the set A for we can always find an element in A corresponding to any element in W and vice versa.

Exercise 2 **a** $B = \{3, 6, 9, \ldots\}$. Map this set into W as we did with A above. Find the elements in B corresponding to:

i 20 ii 35 iii 47 iv 194 in W.

b $C = \{2, 4, 8, 16,\} \ldots$. Copy out this set on a loose sheet of paper leaving plenty of room at the right to extend it. Write down at least the first ten elements. Map the set into the set W as before. Do not forget where you have written this result. You are going to need it again.

c $2+3 = 5$. You will notice that 2, 3 and 5 are elements of W. What are the corresponding images in C? What can we do to the first of these two images to get the third?

d Try the same with $2+4 = 6$; $1+3 = 4$; and two of your own. Does your rule always work? Do you always multiply the images of numbers in W? Could you think of it the other way round: if we multiply two numbers in C, then we add the images in W. Is this true?

Exercise 3 Extend your set C and its index W to at least 17 elements. Use your sets to complete the tables below correctly. Find the easiest way you can (apart from looking at the answers!) and then check afterwards that the multiplications and divisions are in fact correct.

	Numbers in C	Images in W		Numbers in C	Images in W
a	64 × 8 ——	6 + —	**h**	2048 ÷ 16 ——	
b	× 32 ——	2 + —	**i**	4096 ÷ 256 ——	
c	128 × ——	+ 4 —	**j**	32 × 16 ——	
d	512 × 256 ——	+ —	**k**	128 ÷ 8 ——	
e	1024 × 128 ——	—	**l**	8192 ÷ 1024 ——	
f	64 ÷ 8 —— 8	6 3 — 3	**m**	8192 ÷ 32 ——	
g	32 ——	− 2 —	**n**	1024 × 512 ——	

Exercise 4 Here are two sets:

$D = \{3, 9, 27, 81, 243, \ldots\}$

$W = \{1, 2, 3, 4, 5, \ldots\}$

a i What description can you give for D?

ii What values have the following: 3^1, 3^2, 3^3, 3^4, etc.?

iii What have these numbers to do with both D and W?

iv What are the values of 2^1, 2^2, 2^3, 2^4, etc.?

v What have these numbers to do with sets C and W in the last question?

vi When we write a ‘ small number at the top ’, as in 3^5, we can call this number an *index* number. Why?

vii What other names have you met for this particular sort of index number?

b Continue the sets D and W for at least another four terms. Now complete the table below.

	Numbers from D	Images in W
i	243 × 27 ——	
iii	19,683 ÷ 27 ——	

	Numbers from D	Images in W
ii	81 ÷ 27 —	
iv	81 × 27 —	

c Make up six more questions for yourself using a layout like the one in (b) above. Decide where you can leave spaces to be filled (as we did in Exercise 3). Hand the questions to your neighbour to solve with the sets and check his answers yourself by ordinary arithmetic.

Exercise 5 Shorten the following, but leave them in index form.

a $2^4 \times 2^6$

b $2^5 \times 2^3$

c $2^8 \div 2^4$

d $2^9 \div 2^5$

e $2^{17} \div 2^{10}$

f $2^8 \times 2^9$

g $3^5 \times 3^2$

h $3^5 \times 3^7$

i $3^7 \div 3^5$

How are these questions connected with what you have been doing?

Exercise 6 Here is another set with its index set:

$E = \{10, 100, 1000, 10{,}000, \ldots\}$
$W = \{1, 2, 3, 4, \ldots\}$

Do not write out any more members of these sets. You can do the questions without extending your lists.

a Find $10^2 \times 10^4$, $10^6 \div 10^3$, $10^{15} \div 10^9$, $10^4 \times 10^4$ in ordinary numbers.

b Write out 10^{17} in ordinary numbers.

c Put down 1,000,000 in index notation.

d Can you see a rule connecting index notation with the ordinary way of writing the numbers down?

We know that $2^7 = 128$. We can say that 7 is a *logarithm* of 128. We know that $2^4 = 16$. So we say that a logarithm of 16 is 4. But we know that $4^2 = 16$ too. So a logarithm of 16 is 2. It could also be 1 for $16^1 = 16$. We must have a way of knowing which is which. We say that 4 is the logarithm of 16 to the base 2 and 2 is the logarithm of 16 to the base 4.

Exercise 7 Complete the following statements:

a 6 is the logarithm of 64 to the base .

b is the logarithm of 512 to the base 2.

c 10 is the logarithm of to the base 2.

d 4 is the logarithm of to the base 3.

e is the logarithm of 6561 to the base 3.

f 9 is the logarithm of 19,683 to the base .

g is the logarithm of 1000 to the base 10.

h 7 is the logarithm of to the base 10.

i 7 is the logarithm of to the base 2.

j 7 is the logarithm of to the base 3.

k is the logarithm of 100 to the base 10.

l is the logarithm of $10^6 \times 1{,}000{,}000$ to the base 10.

m is the logarithm of 10^{27} to the base 10.

n is the logarithm of $10^6 \times 10^9$ to the base 10.

These sentences are very long and in mathematics we always try to shorten sentences by using symbols. For instance we can rewrite:

'4 is the logarithm of 16 to the base 2'

as

'$4 = \log 16$ (base 2)'.

This can be made even shorter in this way:

'$4 = \log_2 16$'.

Exercise 8 **a** Translate each of the following into words:

i $2 = \log_4 16$
ii $6 = \log_2 64$
iii $5 = \log_3 243$
iv $8 = \log_3 6562$
v $81 = \log_3 4$
vi $12 = \log_{10} 1{,}000{,}000{,}000{,}000$.

b The sentences in (a) are all statements but some of them are false. Which ones are true?

c Rewrite the 14 sentences in Exercise 7 in the new shortened form.

In the work of this section, you have used sets called C, D, E and W. We are now going to draw graphs connected with these sets.

$C = 2, 4, 8, 16, \ldots$

$W = 1, 2, 3, 4, \ldots$

The product set

$$W \times C = \{(1, 2), (1, 4), (1, 8), (1, 16), \ldots (2, 2), (2, 4), (2, 8), \ldots (3, 2), (3, 4) \ldots \text{etc.}\}$$

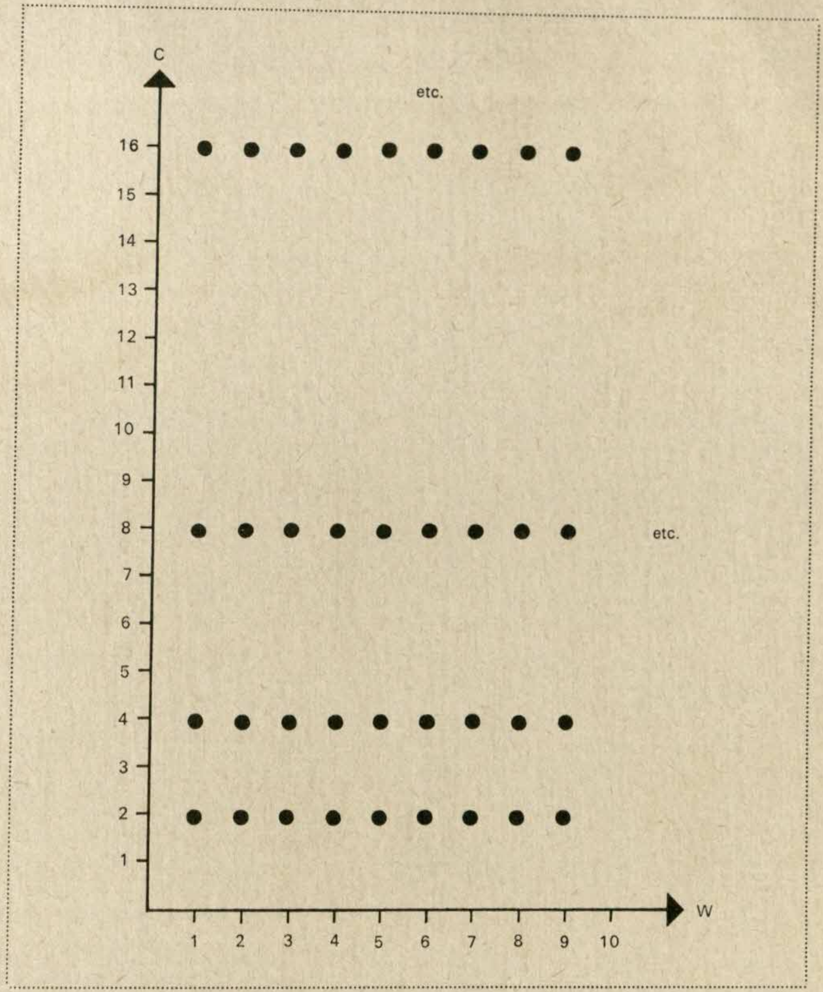
Opposite is a graph of this set of couples.

This is a graph of the whole set of couples which can be made from the two sets by any mapping we care to choose. If we use one particular mapping only, we shall have a subset of the whole set. For instance the one-to-one mapping illustrated here:

$$\begin{array}{l} C = \{2, 4, 8, 16, \ldots\} \\ \quad\;\;\; \updownarrow \; \updownarrow \; \updownarrow \;\; \updownarrow \\ W = \{1, 2, 3, \;\; 4, \ldots\} \end{array}$$

gives us a subset of $W \times C = \{(1, 2), (2, 4), (3, 8), (4, 16), \ldots\}$. There is a graph of this subset on page 224.

The points marked with crosses are the points of the subset. The points marked with dots are the rest of the set. The whole set is sometimes called a *space* and the subset a *subspace*. This sort of space is not the same as the sort of space we have talked about in geometry. We shall see the connexion later.



Exercise 9 **a** Graph the couples you obtain from the following one-to-one mapping of sets D and W:

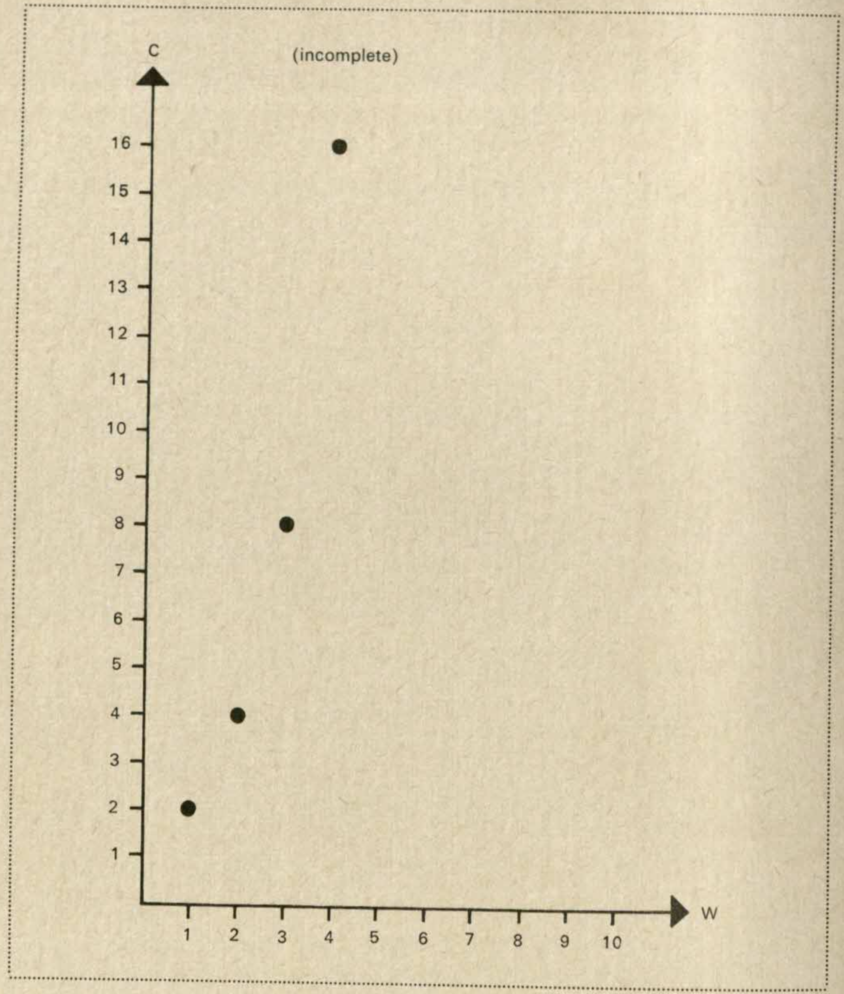
$$\begin{array}{l} D = \{3, 9, 27, 81, 243, \ldots\} \\ \quad\ \ \updownarrow\ \updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \ \updownarrow \\ W = \{1, 2,\ \ 3,\ \ 4,\ \ \ 5, \ldots\} \end{array}$$

Subset of W × D = {(1, 3), etc.}.

Mark with dots the points representing the other couples you can get from other mappings.

b Repeat this for a subset of W × E.

c Do you notice anything about the three graphs that you now have?


C
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W

Section 41 **Areas**

Exercise 1

a What is area? Give examples to explain.

b How do we measure area? Explain how we can find the area of, say, a leaf.

c What is the tessellation which is usually used in saying how big an area is?

d What units would we use to give the area of: (i) a leaf, (ii) a table-top, (iii) a window, (iv) the floor of a room, (v) a small field, (vi) a large field?

e Make up a table of areas showing the number of sq. in. in a sq. ft., sq. ft. in a sq. yd., sq. yds. in an acre, sq. yds. in a sq. mile.

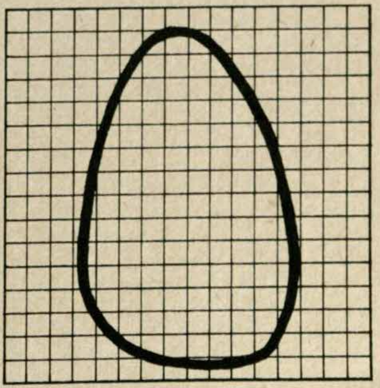
f Work out how many sq. in. there are in a sq. yd. and how many acres in a sq. mile.

g Do only flat things have area?

h Can you think of some things which do not have any area at all? Why do they not have area?

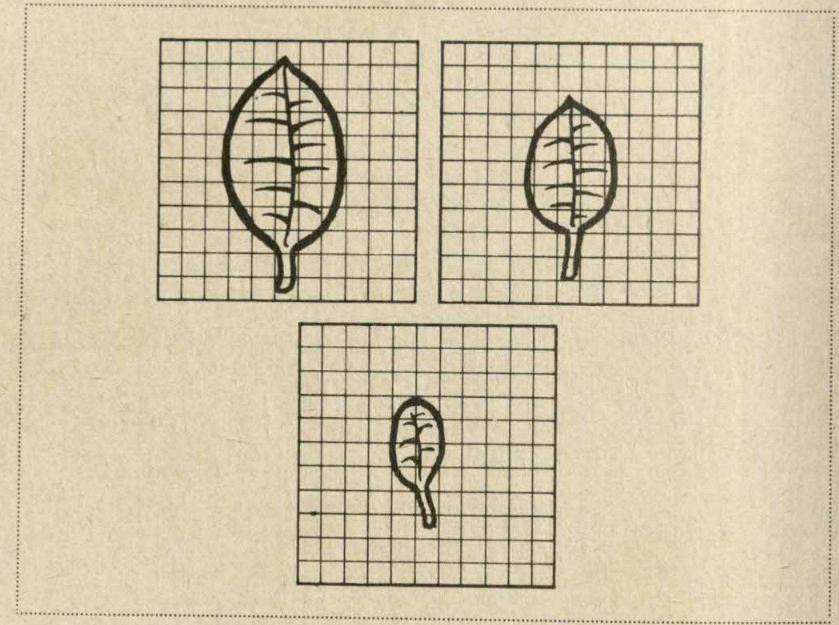
i Can you find the approximate area of any shape?

Joe wondered how big the sole of his shoe was, so he drew round it on a sheet of squared paper. The squares were $\frac{1}{4}$ inch long. His drawing looked like this:

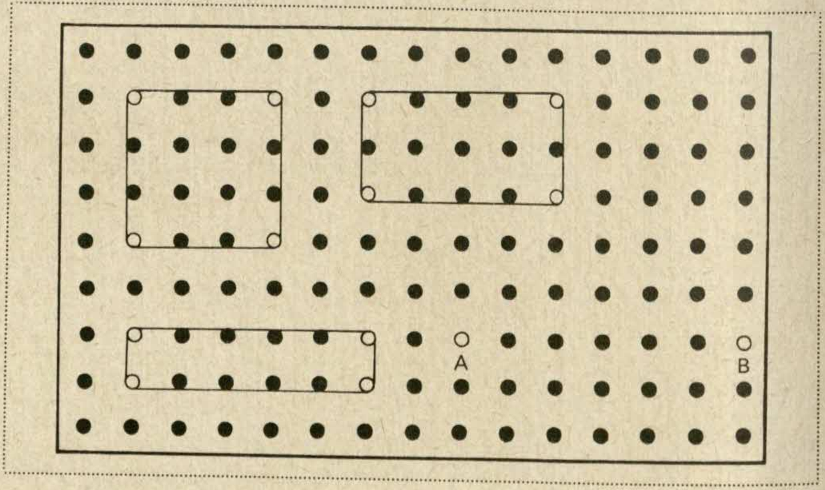


He counted the squares. He wondered if his father's was bigger, so he tried that too.

Jean had a rubber-plant. She wondered what the sizes of the leaves were at various heights above the stem. She drew squares on tracing paper. Each square was $\frac{1}{4}$ in. wide. She put the paper onto each leaf in turn like this:



Jim had some elastic bands and he wondered how big a rectangle he could make with each. He took a piece of peg-board with holes every inch:



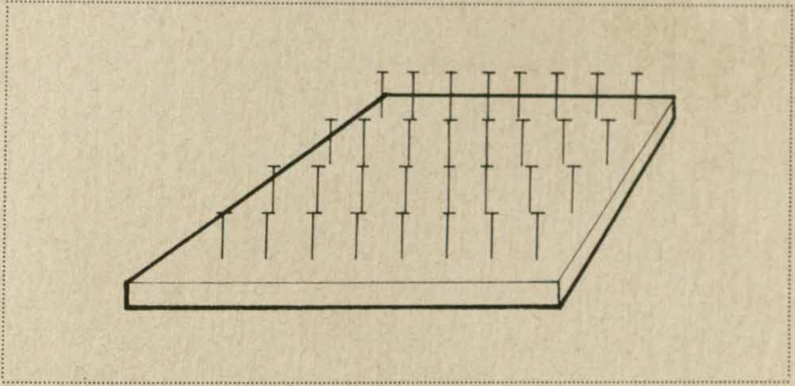
He stretched his bands round pegs which he placed in the holes. He was careful to pull the bands tight but not so tight that he broke them, and he made rectangles as big as he could. He found several different areas with each band. One he tried made the rectangles in the diagram.

Exericse 2 **a** What were the areas of his first three rectangles?

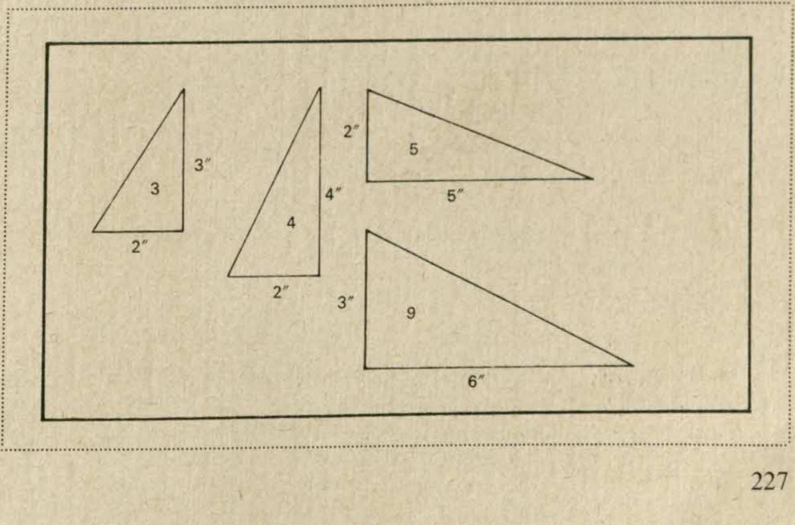
b When he stretched this band tightly round two pegs A and B, how far apart were they?

c Which arrangement of pegs gave him the largest area?

Janet was interested in triangles. She used a board with nails in it 1 in. apart:



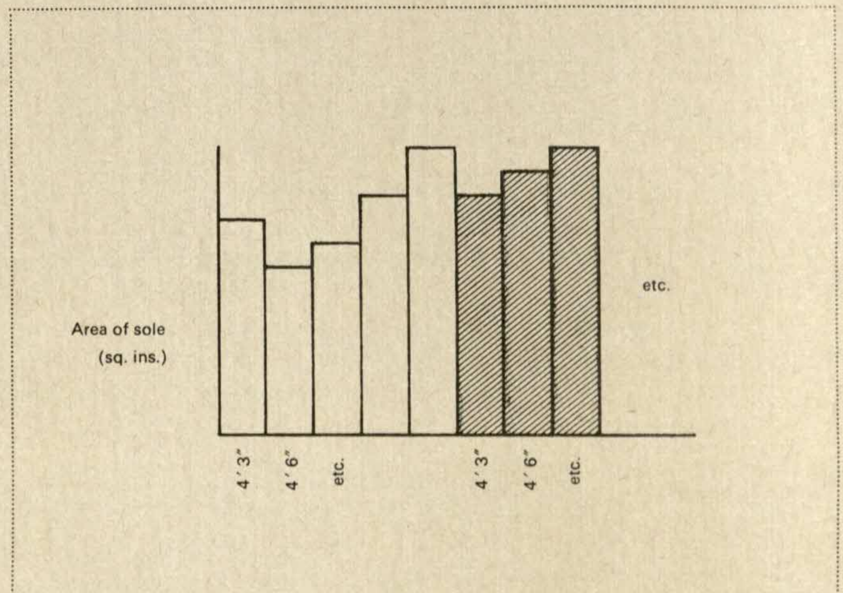
She used elastic bands too and some of her experiments looked like this:



She made sketches of her triangles and wrote inside each the area as we have done, but she forgot to put in the units. What should they have been?

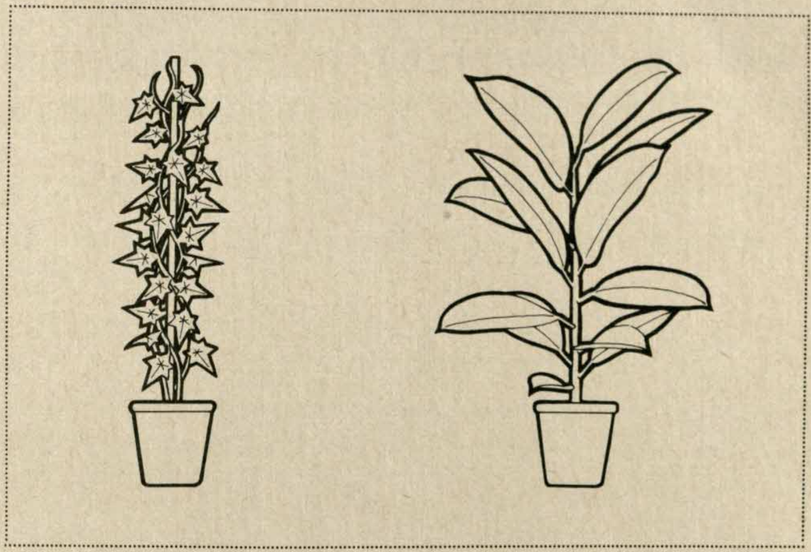
Here are some exercises for you to do:

Exercise 3 Draw round the soles of the shoes of various people. Make sure your paper is big enough to do this properly. If you use squared paper you will not have to draw squares on it afterwards. Record on each sheet the size of shoe and the height of the owner. Say whether it was a man's or a woman's shoe, a girl's or a boy's. If you are working in a group you can all draw a block graph when you have finished. Use four different colours, one for man, another for woman and so on. It may look like this:



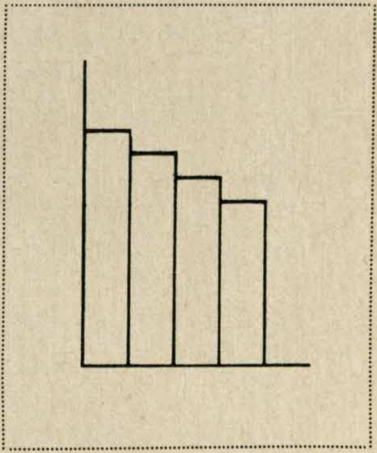
Our sketch is not complete. Do you think that it is right? Explain. Do larger sizes of shoe go with larger areas? Do larger shoes go with taller people?

Exercise 4 Choose a plant, a rubber plant or a climbing plant, for instance. Find the areas at different heights in the way Jean did. Measure the length of stem between the leaves.



Draw a block graph of the distances from the soil and the areas of the leaves.

Is your graph anything like this one?



Exercise 5 Do Jim's experiment for yourself with various sizes of elastic bands. With each band you can make a lot of areas. Which is always the largest? Take a large band which will extend to about 12 in. when it is stretched. Draw sketches of the rectangles which you can get by Jim's method and make a table of the results:

length	breadth	area
11 in.	1 in.	11 sq. in.
10 in.	2 in.	20 sq. in.
9 in.	3 in.	. . . .
8 in.	. . . .	. . . .
. . . .	. . . .	. . . .
. . . .	. . . .	. . . .
5 in.	. . . .	. . . .

Draw a graph of the results. Put area on the vertical axis and length on the horizontal axis.

Jim could not make a rectangle $9\frac{1}{2}$ in. on his board. If there had been holes in the right places, could he have done so? What about all the other sizes between the ones you have done? If you draw a smooth curve through the points on the graph, you will be able to see what the area would have been for any rectangle you could have made.
Now use all your information to find out the following:

a a rule for working out the area of a rectangle,

b two slightly different rules for finding the PERIMETER (that is 'the distance all the way round') of any rectangle,

c the width of the $9\frac{1}{2}$-in. rectangle which Jim might have made,

d by calculation, the area of this rectangle,

e what your graph says is the area of the $9\frac{1}{2}$-in. rectangle.

There is one more experiment you should try with your results. Write down the areas in order, beginning with the smallest and ending with the largest. Write down the first and second differences. We have begun for you here:

Areas 11 20 27 . . . .
lst differences 9 7 . . . .
2nd differences 2 . . . .

What do you notice?

Exercise 6 Without actually making the areas on a peg-board, fill in a table like the one in the last exercise for an elastic band which will extend to 15 in. Draw the graph as before. Can you find from the graph the area of the square which you could make with the band? What do you think would be the area of a rectangle 15 in. long? Does this fit in with the experiment? Find the first and second difference of the areas. Does the same kind of thing happen as before?

Exercise 7 Use your pin-board or draw on squared paper six triangles like Janet's. Do not make your triangles the same sizes as hers. Find out the areas of your triangles. If you used the pin-board draw sketches of the triangles. Write the areas inside each.

Exercise 8 Here are some questions for you to work out : complete the tables below.

a Rectangles :

l	b	A	P
5 in.	7 in.		
16 cm.		64 sq. cm.	
	$2\frac{1}{2}$ in.	15 sq. in.	
57 ft.	12 ft.		
57 ft.	12 in.	sq. ft.	
36 ft.	14 in.	sq. ft.	
	$5\frac{1}{4}$ ft.	$26\frac{1}{4}$ sq. ft.	
5·8 cm.	13·4 cm.		
	13 cm.	170·3 sq. cm.	

b Triangles :

b	h	A
7 cm.	2·5 cm.	
17 in.	$16\frac{1}{2}$ in.	
5 in.	10 in.	
12 cm.	8 cm.	
12 cm.	9 cm.	
16 cm.	15 cm.	
7 in.		28 sq. in.
9 in.		36 sq. in.
9 cm.	7 cm.	
11 in.	13 in.	
14 in.	13 in.	
	5 cm.	25·5 sq. cm.
17 ft.		51 sq. ft.
32 yd.		1 acre

Reinforcement Exercise 6

There are many methods of working out pythagorean triads. Here is one. Fill in the table in accordance with the expressions at the top. The last three terms in any row should form a triad. Check that they do.

The table displays many other patterns. See what you can find out about it. Try any idea you have with new values of *m* and *n* of your own choosing. Why does the method work?

m	n	m^2-n^2	$2mn$	m^2+n^2
2	1			
3	2			
4	3			
5	4			
6	5			
5	2			
4	1			
6	2			
4	3			
12	5			
60	11			
5	4			
13	12			
6	3			
10	6			

Answers to Reinforcement Exercises

Exercise 2
(page 45)

i $5x^3+2x^2+2x+3$
ii $4+4y-4y^2$
iii $5c-2b-2a$
iv $6x^2-2y^2-3y^3-xy$
v a^2-2ab
vi $12xy$
vii $3x+3y$
viii $2x+xy$
ix $4a^2+4ab$
x $2ab+6bc$

Exercise 3
(page 78)

a
i 8
ii 34
iii 19
iv 23
v 17
vi 56
vii 45
viii 7
ix 9
x 5
xi 14
xii 2

b
i $\frac{4}{5}$
ii 4
iii 2
iv $-\frac{7}{6}$
v -6
vi 15
vii $\frac{3}{4}$
viii $\frac{5}{9}$
ix 4
x -2
xi $\frac{6}{17}$
xii $6\frac{1}{12}$

c
i 6
ii 6
iii $\frac{1}{5}$
iv $-\frac{1}{2}$
v $-\frac{1}{7}$
vi $\frac{16}{9}$
vii $\frac{3}{2}$
viii $\frac{17}{5}$
ix -3
x $\frac{8}{3}$
xi 51
xii $\frac{14}{3}$

d
i -1
ii $-\frac{5}{4}$
iii -12
iv -2
v -3
vi -2
vii $\frac{10}{11}$

Exercise 4
(page 183)

i $\begin{pmatrix} 12 & 2 \\ 4 & -4 \end{pmatrix}$

ii $\begin{pmatrix} 1 & 9 \\ 7 & 7 \end{pmatrix}$

iii $x^2-xy-6y^2=(x-3y)(x+2y)$
iv $x(9xy+3y^2-2x^2+5y)$
v x^5
vi $4y^2$
vii $x+y$
viii 16
ix $2{\cdot}73\times10^8$
x 8000

Exercise 5
(page 198)

i ${}^{+}7$
ii ${}^{-}1$
iii ${}^{-}1$
iv ${}^{+}1$
v ${}^{-}7$
vi ${}^{-}1$
vii ${}^{+}1$
viii ${}^{+}1$
ix ${}^{+}7$
x ${}^{-}1$
xi ${}^{+}7$
xii ${}^{-}7$
xiii ${}^{+}7$
xiv ${}^{+}1$
xv ${}^{-}7$
xvi ${}^{-}7$

Symbols

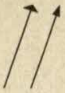
$\cup$	union
$\cap$	intersection
$\{\ \}$, ϕ	the empty set
I, U, E, $\mathscr{E}$	the universal set
$\subset$	is a subset of, is contained in
$\supset$	contains
$\in$	is an element of
$n(A)$	the number of the set A
$\leftrightarrow$	is equivalent to (of sets)
$=$	equals (of sets or numbers)
N	the set of natural numbers
Z	the set of integers
Q	the set of rational numbers
R	the set of real numbers
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	is equivalent to (of statements)

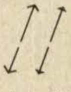
$>$	is greater than
$<$	is less than
$\geqslant$	is greater than or equal to
$\leqslant$	is less than or equal to
$\mid$	such that
$\forall$	for all
$\exists$	there exists (at least one)
$\doteqdot$ $\risingdotseq$ $\simeq$	approximates to
$\sim$	ordered difference
$[\ ]$	the greatest integer less than
$\left(\ \right)$	matrix
$\sqrt{\ }$	square root
$\sqrt[3]{\ }$	cube root
$\uparrow$	is equipollent to
$\parallel$	is parallel to

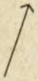
Note

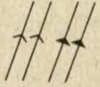
- $\sim$ is sometimes used in mathematics to mean *absolute difference* where order does not count. It is not so used in this book.
- : is used as a punctuation mark in mathematical sentences. Some books use it instead of | meaning *such that.*
- ( ) is used in this book (in conjunction with [ ]) to mean *the smallest integer greater than.* It is not a standard mathematical symbol with this meaning.
- $\mathscr{E}$ is rapidly becoming accepted as the best symbol for *the universal set.*

On diagrams:


denotes *half-lines*


denote *dimensions*


denotes a *vector*


indicate *parallelism*

Other arrows are self explanatory.

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